Generalities

In this preliminary chapter we collect some well known concepts, theorems and techniques which are used several times throughout this thesis.

0.1 The Laplacian Matrix and the Stress Matrix

Let $G = (V, E)$ be a connected simple graph with vertex set $V = \{1, 2, \dots, n\}$.

The *adjacency matrix* of G is the square matrix $A = (a_{ij})$ indexed by $V \times V$, with entries $a_{ij} = 1$ if the vertices i and j are adjacent, and $a_{ij} = 0$ otherwise. Let d_i be the degree of the i-th vertex. Let $D = \mathrm{diag}(d_1, \dots, d_n)$ be the diagonal $V \times V$ matrix with i-th diagonal element equal to d_i, $i = 1, \dots, n$. Then, the *Laplacian matrix* of G is defined to be $L := D - A$. This matrix has wide applications. See [40] for a survey. A *reduced Laplacian matrix* $\bar{L}$, is obtained from L by deleting a certain number of rows and corresponding columns. In the particular case when we delete from L the rows and columns corresponding to the vertices on the outer face of a plane graph G, $\bar{L}$ is known as *stress matrix*.

0.2 Equilibrium Stresses and the Classical Tutte Embedding

We introduce now the notion of *stresses*, which models rubber bands in mathematical terms. Given a graph $G = (V, E)$, to each edge $\{i, j\} \in E$ we assign a weight $\omega_{ij} \in \mathbb{R}$, also known as *stress*, that represents the elasticity constant of the corresponding rubber band. In addition we require the symmetry condition $\omega_{ij} = \omega_{ji}$.

A negative weight means that the edge is pushing on its two endpoints by an equal amount, a positive weight means that the edge is pulling on its endpoints by an equal amount, and zero means that the edge induces no force. The whole stress is denoted by $\omega = (\dots, \omega_{ij}, \dots)$.

Definition 0.1. *Let $G = (V, E)$ be a graph and $\omega \colon E \to \mathbb{R}$ be an assignment of weights to the edges of E. Furthermore, let $\mathbf{p} \colon V \to \mathbb{R}^2$ be an assignment of positions in $\mathbb{R}^2$ for the vertices of G. We say that a vertex $i \in V$ is in* equilibrium *if*

$$\sum_{\{i,j\} \in E} \omega_{ij}(\mathbf{p}_i - \mathbf{p}_j) = 0.$$

Given a planar framework with a convex outer face F, the *interior vertices* and *interior edges* are those which are not incident to F. The vertices and edges of F are called *boundary vertices* and *boundary edges*.

Theorem 0.1 (Tutte's Theorem [48], 1962). *Let $G = (\{1,\ldots,n\}, E)$ be a 3-connected planar graph that has a face $(1,\ldots,k)$ for some $k < n$. Let $\mathbf{p}_1,\ldots,\mathbf{p}_k$ be the vertices (in this order) of a convex k-gon. Let E' be the set of interior edges, and let $\omega\colon E' \to \mathbb{R}^+$ be an assignment of positive weights to the interior edges. Then,*

1. *There are unique equilibrium positions $\mathbf{p}_{k+1},\ldots,\mathbf{p}_n \in \mathbb{R}^2$ for the interior vertices.*
2. *All faces of G are realized as non-overlapping convex polygons.*

We also use the following variant of Tutte's Theorem for non-strictly convex boundaries:

Theorem 0.2. *Let $G = (\{1,\ldots,n\}, E)$ be a 3-connected planar graph that has a face $(1,\ldots,k)$ for some $k < n$. Let $\mathbf{p}_1,\ldots,\mathbf{p}_k$ be the vertices (in this order) of a non-strictly convex k-gon. Three or more boundary points can be on a straight line, but then there can be no edge between them unless they are directly adjacent, i. e., joined by a boundary edge. Let E' be the set of interior edges, and let $\omega\colon E' \to \mathbb{R}^{\geq 0}$ be an assignment of non-negative weights to the interior edges. Then,*

1. *There are unique equilibrium positions $\mathbf{p}_{k+1},\ldots,\mathbf{p}_n \in \mathbb{R}^2$ for the interior vertices.*
2. *All faces of G are realized as non-overlapping convex polygons.*

As in [48], we assume that a 3-connected planar graph $G = (V, E)$ is given with a choice of a peripheral polygon and an assignment of positive stresses to the interior edges.

For any given interior stress, it is easy to find an embedding in which all interior vertices are in equilibrium, by fixing the boundary vertices at some arbitrary location and solving the following system. Choose the coordinates $\mathbf{p}_1,\ldots,\mathbf{p}_k$ of the vertices of the peripheral polygon, in convex position. The graph has $n-k$ interior points, $\mathbf{p}_{k+1},\ldots,\mathbf{p}_n$. We set all interior stresses to 1 (but it works for an arbitrary assignment of positive interior stresses).

Now we impose equilibrium stress: at each interior point forces must add up to zero. According to Definition 0.1, for every interior point $\mathbf{p}_i = (x_i, y_i)$ we have

$$\sum_{\{i,j\}\in E} (x_i - x_j) = 0, \qquad \sum_{\{i,j\}\in E} (y_i - y_j) = 0.$$

Hence we have two separate systems of equations for the x-coordinates and for the y-coordinates, which can be written in matrix form as

$$\bar{L} \cdot \mathbf{x} = \mathbf{b}_\mathbf{x}\,, \qquad \bar{L} \cdot \mathbf{y} = \mathbf{b}_\mathbf{y}\,, \tag{1}$$

where $\mathbf{x} = (x_{k+1}, x_2, \ldots, x_n)^t$ and $\mathbf{y} = (y_{k+1}, y_2, \ldots, y_n)^t$ are the coordinates of the interior points, and $\bar{L}$ is the stress matrix of size $n-k$, obtained from the Laplacian matrix L by deleting the rows and columns corresponding to the boundary vertices.

Let d_i denote the degree of the vertex i. Then the i-th row of $\bar{L}$ contains an entry d_i on the diagonal and an entry -1 at positions j whenever $\{i,j\}$ is an edge of G. All other entries are zero. The independent vectors $\mathbf{b}_\mathbf{x}$ and $\mathbf{b}_\mathbf{y}$ in the system (1) contain the fixed boundary conditions.

The system (1) can be solved uniquely by Theorem 0.1. Hence $\bar{L}$ has rank $n-k$ and the values of x_i and y_i can be uniquely determined by Cramer's rule. For x_i, we have

$$x_i = \det \bar{L}^{(i)} / \det \bar{L},$$

where $\bar{L}^{(i)}$ is obtained from $\bar{L}$ by replacing the i-th column by $\mathbf{b_x}$. The same holds for y_i.

Hence we have computed the coordinates of the interior points. The obtained embedding is also known as *spring embedding*, what can be seen as a system of forces in equilibrium that places every node into the weighted center of gravity of its neighbours, and edges are drawn as straight lines. By Theorem 0.1, this embedding is planar and the interior faces form a proper cell decomposition by convex polygons. An example is illustrated in Figure 4. See [48] for more background information.

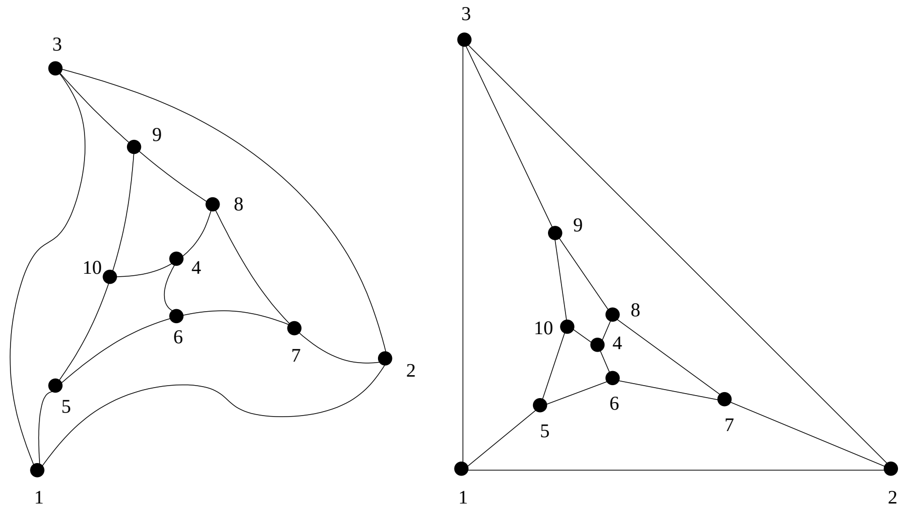

Figure 4: A graph (on the left) and its stressed embedding (with non-uniform stresses, on the right). The boundary vertices are $1, 2, 3$. The interior vertices are $4, 5, 6, 7, 8, 9, 10$.

0.3 From Stressed Configurations to Polyhedra

We summarize here the approach of Richter-Gebert [48] for obtaining the Maxwell-Cremona correspondence for planar configurations.

Let $G[\mathbf{p}] = (\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n)$ be the unique equilibrium configuration given by Tutte's Theorem. We may assume the plane in which the points of $G[\mathbf{p}]$ are located is embedded in $\mathbb{R}^3$ at the plane $z = 1$. Therefore, each $\mathbf{p}_i$ has coordinates $(x_i, y_i, 1)$. Let c_0 be the cell corresponding to the peripheral polygon and let the interior cells c_i be indexed by $i = 1, \ldots, m$. Given an oriented interior edge (b, t) of $G[\mathbf{p}]$, there is, by Tutte's Theorem, a unique adjacent cell L to the left of it, and a unique adjacent cell R to the right of it. An *oriented patch* of $G[\mathbf{p}]$ is an (ordered) tuple $(b, t \mid L, R)$. (The letters are chosen as mnemonics as in [48] for $b =$ *bottom*, $t =$ *top*, $L =$ *left*, $R =$ *right*.)

To each cell c_i we associate a vector $\mathbf{q}_i \in \mathbb{R}^3$ by setting

1. $\mathbf{q}_0 = (0, 0, 0)$;

2. $\mathbf{q}_L = \omega_{bt}(\mathbf{p}_b \times \mathbf{p}_t) + \mathbf{q}_R$ if $(b, t \mid L, R)$ is an oriented patch of $G[\mathbf{p}]$.

The vectors $\mathbf{q}_i$ are computed recursively, choosing a sequence (path) of cells, from $\mathbf{q}_0$ to $\mathbf{q}_i$. Walking along this sequence, $\mathbf{q}_i = \mathbf{q}_L$ is computed from $\mathbf{q}_{i-1} = \mathbf{q}_R$ when we leave the cell

R to enter the cell L, crossing the edge (b,t) (we consider the oriented patch $(b,t \mid L,R)$). It is proved that the vectors $\mathbf{q}_i$ are well defined, that is, that we obtain the same value for $\mathbf{q}_i$ independently from the chosen path.

The *lifting function* or *height function*, is a piecewise linear function h from our configuration $G[\mathbf{p}]$ to $\mathbb{R}$, defined by

$$h(\mathbf{p}_x) = \langle \mathbf{p}_x, \mathbf{q}_i \rangle \qquad \text{if} \qquad \mathbf{p}_x \in c_i\,. \tag{2}$$

It is known that the height function h is well defined, that is, that for adjacent cells c_L and c_R, the functions $\langle \mathbf{p}_x, \mathbf{q}_L \rangle$ and $\langle \mathbf{p}_x, \mathbf{q}_R \rangle$ agree along the common edge, and therefore h defines a unique height for each of the vertices $\mathbf{p}_1, \ldots, \mathbf{p}_n$.

0.4 The Maxwell-Cremona Theorem

The Maxwell-Cremona Theorem [20, 21, 22, 48, 56] is a powerful tool that establishes a bijection between the set of stresses in equilibrium of a configuration in $\mathbb{R}^2$ and the set of three-dimensional polyhedral terrains in $\mathbb{R}^3$ that project onto it.

We state here the one-to-one correspondence given by the Maxwell-Cremona Theorem between positive stresses in equilibrium and convex polytopes projecting onto $G[\mathbf{p}]$. The direction from the set of stresses of the two-dimensional framework to the three-dimensional convex polytope is called *lifting*, and the other direction, from the convex polytope to the set of stresses, is called *projection*.

If a face of a polytope lies on the plane $z = ax + by + c$, we call the vector $\binom{a}{b}$ the *gradient* of the face, and $(a, b, -1)^t$ is its normal vector.

Theorem 0.3 (The Maxwell-Cremona correspondence)**.**

1. *Let $G[\mathbf{p}]$ be a planar framework with a convex outer face F. There is a correspondence between*
   - *(a) positive stresses on the interior edges which are in equilibrium at all interior vertices*
   - *(b) concave piecewise linear liftings of G.*
2. *Let $G[\mathbf{p}]$ be a planar framework with a convex outer face F. There is a correspondence between*
   - *(a) stresses which are positive on all interior edges and negative on all boundary edges and which are in equilibrium at all vertices,*
   - *(b) concave piecewise linear liftings of G such that the boundary edges are horizontal,*
   - *(c) convex polytopes P projecting on G.*
3. *Let $G[\mathbf{p}]$ be a union of two planar frameworks which share a common convex outer face F. (Then G is itself a planar graph.) There is a correspondence between*
   - *(a) stresses which are positive on all interior edges and negative on all boundary edges and which are in equilibrium at all vertices.*
   - *(b) convex polytopes P projecting on G.*

   The boundary of F corresponds to the edges and vertices with vertical supporting planes, and the two planar frameworks correspond to the upper half and the lower half of P.

In all cases, the correspondence satisfies the following relation: If an edge $\{i, j\}$ *with stress* ω_{ij} *separates two faces* f *and* g *with gradients* $\mathbf{p}_f^*$ *and* $\mathbf{p}_g^*$*, then*

$$\mathbf{p}_g^* - \mathbf{p}_f^* = \pm\omega_{ij}(\mathbf{p}_j - \mathbf{p}_i)^\perp . \tag{3}$$

The correspondence is "almost" one-to-one in the following sense. For every lifting or convex polytope, the projection is unique: there is a unique stress which satisfies (3)*. For every stress, the corresponding lifting or convex polytope is unique up to the addition of an affine-linear function, i. e., up to an affine transformation which keeps vertical lines fixed and leaves vertical distances unchanged.*

0.5 The Perron-Frobenius Theorem

Square matrices whose entries are all non-negative have special properties. In this section, we state the well known Perron-Frobenius Theorem [30]. We need a couple of definitions.

A non-negative square matrix A is an *irreducible matrix* if for each entry i, j, there exists $k \geq 1$ such that the (i, j) entry of A^k is strictly positive, or equivalently, if its underlying graph is *strongly connected*.

A non-negative square matrix A is said to be a *primitive matrix* if there exists $k \geq 1$ such that all entries of A^k are strictly positive. A sufficient condition for a non-negative matrix to be primitive is to be an irreducible matrix with at least one positive main diagonal entry.

Theorem 0.4 (Perron-Frobenius Theorem)**.** *Let* A *be a primitive non-negative matrix. Then there is a unique eigenvalue* $\rho(A)$ *of* A *with largest absolute value, and its associated right eigenvector* u *is positive. This vector is the only right non-negative eigenvector. Similarly, there is a unique positive left eigenvector* v*.*

Moreover,

$$\lim_{k\to\infty} [\rho(A)^{-1}A]^k = L > 0 \,,$$

where L *can be computed as* uv^t*, and* u *and* v *are normalized by the condition* $u^tv = 1$*.*

If we start the iteration $\mathbf{x}^{(i+1)} = \mathbf{x}^{(i)}A$ *with any non-negative nonzero vector* $\mathbf{x}^{(0)}$*, the iterated vectors, after normalizing their length to* $\mathbf{x}^{(i)}/\|\mathbf{x}^{(i)}\|$*, converge to* $v/\|v\|$*.*