

Realization and Counting Problems for Planar Structures:
Trees and Linkages, Polytopes and Polyominoes

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*Al meu bebè, que naixerà d'aquí molt poc
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