# Boundary Value Problems for Complex Partial Differential Equations in Fan-shaped Domains

**DISSERTATION**
**des Fachbereichs Mathematik und Informatik**
**der Freien Universität Berlin**
**zur Erlangung des Grades eines**
**Doktors der Naturwissenschaften**

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**October 2010**

**Tag der Disputation: 11. February 2011**

# Contents

# Abstract

In this dissertation, we investigate some boundary value problems for complex partial differential equations in fan-shaped domains. First of all, we establish the Schwarz-Poisson representation in fan-shaped domains with angle $\pi/n$ $(n \in \mathbb{N})$ by the reflection method, and study the corresponding Schwarz and Dirichlet problems respectively. Further, the Schwarz-Poisson formula is extended to the general fan-shaped domains with angle $\pi/\alpha$ $(\alpha \geq 1/2)$ by proper conformal mappings, and then the Schwarz and Dirichlet problems for the Cauchy-Riemann equation are solved. Next, we also establish a bridge between the unit disc and the fan-shaped domain with $\alpha = 1/2$, and the Schwarz-Poisson formula for the unit disc is derived from the Schwarz-Poisson formula for $\alpha = 1/2$.

Then, we firstly obtain a harmonic Green function and a harmonic Neumann function in the fan-shaped domain with angle $\pi/\alpha$ $(\alpha \geq 1/2)$, and then investigate the Dirichlet and Neumann problems for the Poisson equation. In particular, the outward normal derivative at the three corner points is properly defined. Next, a biharmonic Green function, a biharmonic Neumann function, a triharmonic Green function, a triharmonic Neumann function and a tetra-harmonic Green function are constructed for the fan-shaped domain with angle $\pi/n$ $(n \in \mathbb{N})$ in explicit form respectively. Moreover, we give the process of constructing a tetra-harmonic Neumann function and the expression of the tetra-harmonic Neumann function with integral representation. Accordingly, the Dirichlet and Neumann problems are discussed.

Finally, we establish the iterated expressions and the solvability conditions of polyharmonic Dirichlet and Neumann problems for the higher order Poisson equation in the fan-shaped domain with angle $\pi/n$ $(n \in \mathbb{N})$ respectively. In the meantime, the boundary behavior of polyharmonic Green and polyharmonic Neumann functions by convolution are discussed in detail. Besides, in the Appendix, the tetra-harmonic Green function and the triharmonic Neumann function for

the unit disc are constructed in explicit form.

**Keywords:** Schwarz-Poisson representation, polyharmonic Green function, polyharmonic Neumann function, Schwarz problem, Dirichlet problem, Neumann problem.

# Acknowledgements

I consider myself very fortunate that many people offer me help and give me much motivation, so that I can complete my thesis.

First and foremost, I would like to express my deepest gratitude to Prof. Dr. Heinrich Begehr for giving me the chance to study at this institute. He is very kind and warm-hearted. He spent a lot of care in instructing and revising my papers, and also made great contribution to my extension and to the financial support for my Ph.D project. He always gave me much help with great patience whenever I needed help. For all this and much else besides, I offer him my deep gratitude.

Next I am very grateful to Prof. Dr. Jinyuan Du for his encouragement and assistance in pursuing my study. He is strict and very kind to me. I appreciate him very much for all his instruction and help, which always encourages me to make progress in mathematics. I am also thankful to Prof. Dr. Alexander Schmitt for his kindness and spending time in assessing the thesis.

Then, I owe deep thanks to Dr. Yufeng Wang for his many suggestions and aids both in my study and life in Berlin. I am very thankful to Dr. Zhongxiang Zhang for his much help, Dr. Zhihua Du for his kindness and the China Scholarship Council for the financial support. I also appreciate the help from Ms Caroline Neumann, the Center for International Cooperation in FU Berlin and the support from the STIBET-Program (DAAD).

Finally, I want to express my special thanks to my parents, sisters and friends for their selfless love and encouragement. They always gave me the biggest understanding and tolerance when I was stressful, which helps me to overcome many difficulties.

# Chapter 1

# Introduction

Complex analysis is a comparatively active branch in mathematics which has grown significantly. In particular, the investigation of boundary value problems possesses both theoretical and applicable values of importance to many fields, such as electricity and magnetism, hydrodynamics, elasticity theory, shell theory, quantum mechanics, medical imaging, etc. In recent years, many investigators have made great contribution to boundary value problems for complex partial differential equations. Numerous results are achieved, which rapidly enrich the development of generalized analytic functions, boundary value problems, Riemann-Hilbert analysis, mathematical physics and so on, reference to [6, 31, 32, 37, 45, 50, 51, 55, 56].

The classical boundary value problems initiated by B.Riemann and D.Hilbert are the Riemann and the Riemann-Hilbert problems [46, 38]. The theory of boundary value problems for analytic functions is extended to many branches. Analytic functions are in close connection with the Cauchy-Riemann operator $\partial_{\overline{z}}$. Then one aspect is to investigate boundary value problems for different kinds of functions and the functions satisfying particular complex differential equations, e.g. generalized analytic functions, functions with several variables, functions in Hardy space, functions satisfying the Cauchy-Riemann equation, the Beltrami equation, the generalized Poisson equation, even the higher order complex differential equations, reference to [3, 4, 5, 6, 48, 50, 56]. In particular, great interest has arisen for polyanalytic and polyharmonic equations, see [15, 22, 24, 39, 40]. On the other hand, various types of conditions imposed on the boundary lead to different boundary value problems, such as the Riemann, the Hilbert, the Dirichlet, the Schwarz, the Neumann, the Hasemann, the Robin boundary value problems [18, 22, 33, 35, 39, 43]. Moreover, besides the study in the classical unit disc, much attention has been paid to boundary value problems in some particular domains, for example, a half unit disc, a triangle, a fan-shaped domain, the upper half plane, a quarter plane, a circular ring and a half circular ring [5, 19, 26, 36, 49, 54, 60]. Also, some investigators have extended boundary value problems to higher dimensional spaces, such as a polydisc, a sphere and other torus related domains, reference to [21, 42, 44].

Generally speaking, the fundamental tools for solving boundary value problems are the Gauss theorem and the Cauchy-Pompeiu formula. Besides, the higher order Cauchy-Pompeiu operators $T_{m,n}$, due to H. Begehr and G. Hile [20], establish a bridge for boundary value problems between the homogeneous and the inhomogeneous complex partial differential equations.

As is well known, Green, Neumann and Robin functions are three useful fundamental solutions for certain boundary value problems via integral representation formulas. Especially, in order to solve some polyharmonic Dirichlet and Neumann problems, certain polyharmonic Green and polyharmonic Neumann functions need to be studied. In fact, there are several different kinds of polyharmonic Green functions. Convoluting the harmonic Green function with itself consecutively leads to an iterated polyharmonic Green function, which can be used to solve an iterated Dirichlet problem for the higher order Poisson equation. In addition, different from the above polyharmonic Green functions, polyharmonic Green-Almansi functions are firstly introduced for the unit disc by Almansi [1], which also give rise to some particular polyharmonic Dirichlet problems. Similarly, convoluting the harmonic Neumann function with itself consecutively results in an iterated polyharmonic Neumann function. Besides, iteration of the harmonic Green, Neumann and Robin functions pairwise leads to different hybrid biharmonic Green functions due to H. Begehr [10, 11]. Furthermore, convoluting the iterated polyharmonic Green functions with the polyharmonic Green-Almansi functions also gives a variety of hybrid polyharmonic Green functions [9, 12, 28].

However, it should be noted that the expressions of the polyharmonic Green and Neumann functions by convolution are not easily constructed in explicit form even in the classical unit disc, although the iterated polyharmonic Dirichlet and

Neumann problems can be solved by iterated forms.

Many results have been obtained for boundary value problems of complex partial differential equations in some particular domains. In the unit disc, basic boundary value problems (Schwarz, Dirichlet, Neumann, Robin problems) for the Cauchy-Riemann equation are studied and several hybrid biharmonic functions are given explicitly by convoluting the harmonic ones [7, 12, 25]. Moreover, polyharmonic Poisson kernels for the higher order Poisson equation are constructed in a complicated form using vertical sums [16, 34], but the iterated polyharmonic (m-harmonic)Green and Neumann functions by convolution are established only up to $m = 3$ [29]. For the upper half plane [17, 36], polyharmonic Green-Almansi functions are given explicitly to solve the related polyharmonic Dirichlet problem. And in the circular ring [52, 53, 54], the existing results mainly include solving four fundamental boundary value problems for the Cauchy-Riemann equation, and then establishing the harmonic Green, Neumann, Robin functions as well as the biharmonic Green function in detail. Besides, the investigation for the quarter plane is just at the beginning with basic boundary value problems for the Cauchy-Riemann equation solved [2, 19]. As to in the higher dimensional spaces, the related results can be viewed in [13, 14, 21, 41, 42, 44]. Especially, the Schwarz problem for the Cauchy-Riemann equation, the harmonic Green and Neumann functions are studied in half disc and half ring [26]. Also some results are achieved in fan-shaped domains [57, 58, 59].

In this thesis, we systematically investigate some boundary value problems in fan-shaped domains. First of all, the Schwarz-Poisson representation formula is obtained in fan-shaped domains with angle $\pi/n$ ($n \in \mathbb{N}$) by the reflection method, as well as the Schwarz and Dirichlet problems are discussed for the Cauchy-Riemann equation. For the general fan-shaped domains with angle $\pi/\alpha$ ($\alpha \geq 1/2$), the Schwarz-Poisson representation is established by a proper conformal mapping, and then some boundary value problems for the Cauchy-Riemann equation are investigated. In particular, the boundary behavior at the three corner points are discussed in detail for the above two kinds of domains. Next, the situation for $\alpha = 1/2$ is especially investigated. Furthermore, we develop a bridge between the unit disc and the fan-shaped domain with $\alpha = 1/2$, and then the Schwarz-Poisson formula for the unit disc can be derived from the Schwarz-Poisson formula for the fan-shaped domain with angle $2\pi$ ($\alpha = 1/2$).

Next, the harmonic Green and the harmonic Neumann functions are constructed in the fan-shaped domain with angle $\pi/\alpha$ ($\alpha \geq 1/2$). What is more, the outward normal derivatives at the three corner points are introduced properly and the corresponding Dirichlet and Neumann problems for the Poisson equation are studied. As we know, the construction of polyharmonic Green and polyharmonic Neumann functions of arbitrary order $m$ in explicit form is a demanding and complicated procedure. Here the biharmonic Green, the biharmonic Neumann, the tri-harmonic Green, the tri-harmonic Neumann, and the tetra-harmonic Green functions are established explicitly for the fan-shaped domain with angle $\pi/n$ ($n \in \mathbb{N}$), by means of a series of proper polyharmonic functions. Then we also give the construction process for the tetra-harmonic Neumann function in detail and obtain the expression of the tetra-harmonic Neumann function with integral representation. Accordingly, the Dirichlet and Neumann problems are investigated respectively.

Finally, even though the explicit expressions for polyharmonic Green and Neumann functions are unknown except for the above lower order ones, we still establish the inductive expressions of solutions and solvability conditions for the iterated polyharmonic Dirichlet and polyharmonic Neumann problems in the fan-shaped domain with angle $\pi/n$ ($n \in \mathbb{N}$). At the same time, the recursive expressions of polyharmonic Green and polyharmonic Neumann functions are given by convolution, and their boundary behaviors are investigated in detail. Besides, in the Appendix, the tetra-harmonic Green function and the tri-harmonic Neumann function for the unit disc are constructed in explicit forms respectively.

# Chapter 2

# Boundary Value Problems for the Inhomogeneous Cauchy-Riemann Equation

In this Chapter, the Schwarz-Poisson representation formulas are obtained in fan-shaped domains with angle $\pi/n$ ($n \in \mathbb{N}$) and $\pi/\alpha$ ($\alpha \geq 1/2$) respectively, and then the solutions and solvability conditions for the Schwarz and Dirichlet problems are given explicitly.

## 2.1 Preliminaries

Let $\mathbb{C}$ be the complex plane and the variable $z = x + i\, y,\ x,\ y \in \mathbb{R}$. Introducing the complex partial derivatives,

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \qquad \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

If a continuously differentiable function $w(z)$ satisfies the following homogeneous Cauchy-Riemann equation

$$\partial_{\overline{z}} w(z) = 0,$$

then $w(z)$ is analytic.

The main tools for solving boundary value problems of complex differential equations are the Gauss theorem and the Cauchy-Pompeiu representation.

**Theorem 2.1.1.** [6, 7] (Gauss Theorem) *Let $D \subset \mathbb{C}$ be a regular domain, $w \in C^1(D,\mathbb{C}) \cap C(\overline{D},\mathbb{C}),\ z = x + i\, y$, then,*

$$\int_D w_{\overline{z}}(z)\mathrm{d}x\mathrm{d}y = \frac{1}{2i}\int_{\partial D} w(z)\mathrm{d}z, \tag{2.1}$$

$$\int_D w_z(z)\mathrm{d}x\mathrm{d}y = -\frac{1}{2i}\int_{\partial D} w(z)\mathrm{d}\overline{z}. \tag{2.2}$$

The above Gauss theorem leads to a generalization of the Cauchy representation for analytic functions, that is the so-called Cauchy-Pompeiu formula.

**Theorem 2.1.2.** [6, 7] (Cauchy-Pompeiu representation) *Any* $w \in C^1(D;\mathbb{C}) \cap C(\overline{D};\mathbb{C})$ *for a regular complex domain* $D \subset \mathbb{C}$ *can be represented as*

$$\frac{1}{2\pi i}\int\limits_{\partial D} w(\zeta)\frac{\mathrm{d}\zeta}{\zeta - z} - \frac{1}{\pi}\int\limits_{D} w_{\overline{\zeta}}(\zeta)\frac{\mathrm{d}\xi\mathrm{d}\eta}{\zeta - z} = \begin{cases} w(z), & z \in D, \\ 0, & z \notin \overline{D}, \end{cases} \tag{2.3}$$

*and*

$$-\frac{1}{2\pi i}\int\limits_{\partial D} w(\zeta)\frac{\mathrm{d}\overline{\zeta}}{\overline{\zeta - z}} - \frac{1}{\pi}\int\limits_{D} w_{\zeta}(\zeta)\frac{\mathrm{d}\xi\mathrm{d}\eta}{\overline{\zeta - z}} = \begin{cases} w(z), & z \in D, \\ 0, & z \notin \overline{D}. \end{cases} \tag{2.4}$$

The Pompeiu operator

$$T[f](z) = -\frac{1}{\pi}\int\limits_{D} f(\zeta)\frac{\mathrm{d}\xi\mathrm{d}\eta}{\zeta - z}, \quad f \in L_1(\overline{D};\mathbb{C}),\ z \in D \tag{2.5}$$

studied in detail by I.N.Vekua [50] plays a critical role in treating boundary value problems for the inhomogeneous Cauchy-Riemann equation. It has some important properties.

**Theorem 2.1.3.** [6, 50] *Let* $D \subset \mathbb{C}$ *be a bounded domain, then for* $f \in L_p(\overline{D};\mathbb{C})$, $p > 2$, $T$ *is a completely continuous linear operator from* $L_p(\overline{D};\mathbb{C})$ *into* $C^{\alpha_0}(\mathbb{C})$ *with* $\alpha_0 = \dfrac{p-2}{p}$.

**Theorem 2.1.4.** [6, 50] *If* $f \in L_1(\overline{D})$, *then for all* $\varphi \in C_0^1(D)$

$$\int\limits_{D} T[f](z)\varphi_{\overline{z}}(z)\mathrm{d}x\mathrm{d}y + \int\limits_{D} f(z)\varphi(z)\mathrm{d}x\mathrm{d}y = 0.$$

**Remark 2.1.1.** Theorem 2.1.4 implies that for $f \in L_1(\overline{D})$, $T[f]$ is differentiable with respect to $\overline{z}$ in weak sense with $\dfrac{\partial T[f]}{\partial \overline{z}} = f$.

The Poisson kernels for the unit disc $\mathbb{D} = \{z \in \mathbb{C} : \ |z| < 1\}$ and the upper half plane $\mathbb{H}^+ = \{z : \ \mathrm{Im} z > 0\}$ are, respectively,

$$\frac{\zeta}{\zeta - z} + \frac{\overline{\zeta}}{\overline{\zeta - z}} - 1,\ z \in \mathbb{D},\ \zeta \in \partial\mathbb{D},$$

and

$$\frac{1}{\zeta - z} - \frac{1}{\zeta - \overline{z}}, \quad z \in \mathbb{H}^+, \ \zeta \in \partial\mathbb{H}^+.$$

Then for $\gamma_1 \in C(\partial\mathbb{D}, \mathbb{R})$, $\gamma_2 \in C(\partial\mathbb{H}^+, \mathbb{R})$ [47, 36],

$$\lim_{\substack{z\to t,\ |z|<1 \\ t\in\partial\mathbb{D}}} \frac{1}{2\pi i} \int\limits_{|\zeta|=1} \gamma_1(\zeta) \left[\frac{\zeta}{\zeta - z} + \frac{\overline{\zeta}}{\overline{\zeta - z}} - 1\right] \frac{\mathrm{d}\zeta}{\zeta} = \gamma_1(t).$$

and

$$\lim_{\substack{z\to t,\ z\in\mathbb{H}^+ \\ t\in\partial\mathbb{H}^+}} \frac{1}{2\pi i} \int\limits_{-\infty}^{+\infty} \gamma_2(\zeta) \left[\frac{1}{\zeta - z} - \frac{1}{\zeta - \overline{z}}\right] \mathrm{d}\zeta = \gamma_2(t).$$

## 2.2 Schwarz Problem with Angle $\pi/n$ ($n \in \mathbb{N}$)

Let $\Omega$ be a fan-shaped domain in the complex plane $\mathbb{C}$ defined by

$$\Omega = \left\{z \in \mathbb{C} :\ |z| < 1,\ 0 < \arg z < \frac{\pi}{n}\right\}. \tag{2.6}$$

$0, 1, \omega = e^{i\theta}$ are three corner points of the domain $\Omega$ and the oriented circular arc $L$ is parameterized by

$$L :\ \tau \longmapsto e^{i\tau}, \quad \tau \in [0,\ \theta],$$

where $\theta = \frac{\pi}{n}$. The boundary $\partial\Omega = [0, 1] \cup L \cup [\omega, 0]$ is oriented counter-clockwise.

In what follows, we always regard $n$ as a fixed positive integer, $\theta = \dfrac{\pi}{n}$ and $\omega = e^{i\theta}$.

By rotations, we define some domains

$$\Omega_k \equiv \omega^{2k}\Omega = \{\omega^{2k} z :\ z \in \Omega\}, \quad k = 0, 1, \cdots, n-1, \tag{2.7}$$

where $\Omega_0 = \Omega$ is the sector defined by (2.6). By reflections on the real axis, we define

$$E_k = \{\overline{z} :\ z \in \Omega_k\}, \quad k = 0, 1, \cdots, n-1. \tag{2.8}$$

Besides, by reflections on the unit circumference, define

$$\mathcal{D}_k = \{\overline{z}^{-1} :\ z \in \Omega_k\}, \quad \mathcal{E}_k = \{\overline{z}^{-1} :\ z \in E_k\}, \quad k = 0, 1, \cdots, n-1. \tag{2.9}$$

Obviously, $\Omega_k, \mathcal{D}_k, E_k, \mathcal{E}_k,\ k = 0, 1, \cdots, n-1$ are disjoint domains and

$$\mathbb{C} = \bigcup_{k=0}^{n-1} \left(\overline{\Omega}_k \cup \overline{\mathcal{D}}_k \cup \overline{E}_k \cup \overline{\mathcal{E}}_k\right). \tag{2.10}$$

Moreover,

$$\bigcup_{k=0}^{n-1}(\overline{\Omega}_k \cup \overline{E}_k) = \{z \in \mathbb{C} :\ |z| \leq 1\},\quad \bigcup_{k=0}^{n-1}(\overline{\mathcal{D}}_k \cup \overline{\mathcal{E}}_k) = \{z \in \mathbb{C} :\ |z| \geq 1\}.$$

Obviously, the following lemma holds.

**Lemma 2.2.1.** *If $z \in \Omega$, then $z\omega^{2k} \in \Omega_k$, $\overline{z}\omega^{-2k} \in E_k$, $\overline{z}^{-1}\omega^{2k} \in \mathcal{D}_k$ and $z^{-1}\omega^{-2k} \in \mathcal{E}_k$ for $k = 0, 1, \cdots, n-1$, where $\Omega_k, E_k, \mathcal{D}_k, \mathcal{E}_k$ are defined by* (2.7)-(2.9)*, respectively.*

### 2.2.1 Schwarz-Poisson Representation

To solve the Schwarz problem, the Schwarz-Poisson formula is derived from the Cauchy-Pompeiu formula.

**Theorem 2.2.1.** *Any $w \in C^1(\Omega;\mathbb{C}) \cap C(\overline{\Omega};\mathbb{C})$ can be represented as*

$$\begin{aligned} w(z) = &\frac{1}{2\pi i}\int_{\partial\Omega} w(\zeta)\sum_{k=0}^{n-1}\left(\frac{1}{\zeta - z\omega^{2k}} - \frac{z}{\omega^{2k} - z\zeta}\right)\mathrm{d}\zeta \\ &-\frac{1}{\pi}\int_{\Omega} w_{\overline{\zeta}}(\zeta)\sum_{k=0}^{n-1}\left(\frac{1}{\zeta - z\omega^{2k}} - \frac{z}{\omega^{2k} - z\zeta}\right)\mathrm{d}\xi\mathrm{d}\eta,\ z \in \Omega \end{aligned} \tag{2.11}$$

*and for $z \in \Omega$*

$$\begin{aligned} w(z) \ =& \frac{1}{2\pi i}\int_{L} \mathrm{Re}w(\zeta)\sum_{k=0}^{n-1}\left(\frac{\zeta + z\omega^{2k}}{\zeta - z\omega^{2k}} - \frac{\overline{\zeta} + z\omega^{-2k}}{\overline{\zeta} - z\omega^{-2k}}\right)\frac{\mathrm{d}\zeta}{\zeta} + \frac{n}{\pi}\int_{L}\frac{\mathrm{Im}w(\zeta)}{\zeta}\mathrm{d}\zeta \\ &+\frac{1}{\pi i}\int_{[\omega,0]\cup[0,1]} \mathrm{Re}w(\zeta)\sum_{k=0}^{n-1}\left(\frac{1}{\zeta - z\omega^{2k}} - \frac{z}{\omega^{2k} - z\zeta}\right)\mathrm{d}\zeta \\ &-\frac{1}{\pi}\int_{\Omega}\left[w_{\overline{\zeta}}(\zeta)\sum_{k=0}^{n-1}\left(\frac{1}{\zeta - z\omega^{2k}} - \frac{z}{\omega^{2k} - z\zeta}\right)\right. \\ &\left. -\overline{w_{\overline{\zeta}}(\zeta)}\sum_{k=0}^{n-1}\left(\frac{1}{\overline{\zeta} - z\omega^{2k}} - \frac{z}{\omega^{2k} - z\overline{\zeta}}\right)\right]\mathrm{d}\xi\mathrm{d}\eta. \end{aligned} \tag{2.12}$$

**Proof:** From Lemma 2.2.1 and Theorem 2.1.2,

$$
\begin{aligned}
w(z) = \sum_{k=0}^{n-1} \Bigg\{ \frac{1}{2\pi i} \int\limits_{L} \frac{w(\zeta)}{\zeta - z\omega^{2k}} \mathrm{d}\zeta + \frac{1}{2\pi i} \int\limits_{[\omega,0]\cup[0,1]} \frac{w(\zeta)}{\zeta - z\omega^{2k}} \mathrm{d}\zeta \\
- \frac{1}{\pi} \int\limits_{\Omega} \frac{w_{\bar\zeta}(\zeta)}{\zeta - z\omega^{2k}} \mathrm{d}\xi \mathrm{d}\eta \Bigg\}, \quad z \in \Omega,
\end{aligned}
\tag{2.13}
$$

$$
\begin{aligned}
0 = \sum_{k=0}^{n-1} \Bigg\{ \frac{1}{2\pi i} \int\limits_{L} \frac{z w(\zeta)}{z\zeta - \omega^{2k}} \mathrm{d}\zeta + \frac{1}{2\pi i} \int\limits_{[\omega,0]\cup[0,1]} \frac{z w(\zeta)}{z\zeta - \omega^{2k}} \mathrm{d}\zeta \\
- \frac{1}{\pi} \int\limits_{\Omega} \frac{z w_{\bar\zeta}(\zeta)}{z\zeta - \omega^{2k}} \mathrm{d}\xi \mathrm{d}\eta \Bigg\}, \quad z \in \Omega,
\end{aligned}
\tag{2.14}
$$

$$
\begin{aligned}
0 = \sum_{k=0}^{n-1} \Bigg\{ \frac{1}{2\pi i} \int\limits_{L} \frac{w(\zeta)}{\zeta - \overline{z}\omega^{-2k}} \mathrm{d}\zeta + \frac{1}{2\pi i} \int\limits_{[\omega,0]\cup[0,1]} \frac{w(\zeta)}{\zeta - \overline{z}\omega^{-2k}} \mathrm{d}\zeta \\
- \frac{1}{\pi} \int\limits_{\Omega} \frac{w_{\bar\zeta}(\zeta)}{\zeta - \overline{z}\omega^{-2k}} \mathrm{d}\xi \mathrm{d}\eta \Bigg\}, \quad z \in \Omega,
\end{aligned}
\tag{2.15}
$$

and

$$
\begin{aligned}
0 = \sum_{k=0}^{n-1} \Bigg\{ \frac{1}{2\pi i} \int\limits_{L} \frac{\overline{z} w(\zeta)}{\overline{z}\zeta - \omega^{-2k}} \mathrm{d}\zeta + \frac{1}{2\pi i} \int\limits_{[\omega,0]\cup[0,1]} \frac{\overline{z} w(\zeta)}{\overline{z}\zeta - \omega^{-2k}} \mathrm{d}\zeta \\
- \frac{1}{\pi} \int\limits_{\Omega} \frac{\overline{z} w_{\bar\zeta}(\zeta)}{\overline{z}\zeta - \omega^{-2k}} \mathrm{d}\xi \mathrm{d}\eta \Bigg\}, \quad z \in \Omega.
\end{aligned}
\tag{2.16}
$$

Clearly, adding (2.13) and (2.14) leads to the validity of (2.11).

Taking complex conjugation on both sides of (2.15) and (2.16), respectively, gives

$$
\begin{aligned}
0 = \sum_{k=0}^{n-1} \Bigg\{ \frac{1}{2\pi i} \int\limits_{L} \frac{\overline{\zeta}\ \overline{w(\zeta)}}{\overline{\zeta} - z\omega^{2k}} \frac{\mathrm{d}\zeta}{\zeta} - \frac{1}{2\pi i} \int\limits_{0}^{1} \frac{\overline{w(\zeta)}}{\zeta - z\omega^{2k}} \mathrm{d}\zeta \\
- \frac{1}{2\pi i} \int\limits_{[\omega,0]} \frac{\overline{w(\zeta)}}{\zeta - z\omega^{2k+2}} \mathrm{d}\zeta - \frac{1}{\pi} \int\limits_{\Omega} \frac{\overline{w_{\bar\zeta}(\zeta)}}{\overline{\zeta} - z\omega^{2k}} \mathrm{d}\xi \mathrm{d}\eta \Bigg\}, \quad z \in \Omega
\end{aligned}
\tag{2.17}
$$

and

$$
\begin{aligned}
0 \ &= \sum_{k=0}^{n-1}\left\{\frac{1}{2\pi i}\int_{L}\frac{z\overline{\zeta}\ \overline{w(\zeta)}}{z\overline{\zeta}-\omega^{2k}}\frac{\mathrm{d}\zeta}{\zeta}-\frac{1}{2\pi i}\int_{0}^{1}\frac{z\overline{w(\zeta)}}{z\zeta-\omega^{2k}}\mathrm{d}\zeta\right.\\
&\left.-\frac{1}{2\pi i}\int_{[\omega,0]}\frac{z\overline{w(\zeta)}}{z\zeta-\omega^{2k+2}}\mathrm{d}\zeta-\frac{1}{\pi}\int_{\Omega}\frac{z\overline{w_{\bar{\zeta}}(\zeta)}}{z\overline{\zeta}-\omega^{2k}}\mathrm{d}\xi\mathrm{d}\eta\right\},\quad z\in\Omega.
\end{aligned}
\tag{2.18}
$$

Obviously, (2.17) and (2.18) can be, respectively, rewritten as

$$
\begin{aligned}
0 = \sum_{k=0}^{n-1}&\left\{\frac{1}{2\pi i}\int_{L}\frac{\overline{\zeta}\ \overline{w(\zeta)}}{\overline{\zeta}-z\omega^{2k}}\frac{\mathrm{d}\zeta}{\zeta}-\frac{1}{2\pi i}\int_{[\omega,0]\cup[0,1]}\frac{\overline{w(\zeta)}}{\zeta-z\omega^{2k}}\mathrm{d}\zeta\right.\\
&\left.-\frac{1}{\pi}\int_{\Omega}\frac{\overline{w_{\bar{\zeta}}(\zeta)}}{\overline{\zeta}-z\omega^{2k}}\mathrm{d}\xi\mathrm{d}\eta\right\},\quad z\in\Omega,
\end{aligned}
\tag{2.19}
$$

and

$$
\begin{aligned}
0 = \sum_{k=0}^{n-1}&\left\{\frac{1}{2\pi i}\int_{L}\frac{z\overline{\zeta}\ \overline{w(\zeta)}}{z\overline{\zeta}-\omega^{2k}}\frac{\mathrm{d}\zeta}{\zeta}-\frac{1}{2\pi i}\int_{[\omega,0]\cup[0,1]}\frac{z\overline{w(\zeta)}}{z\zeta-\omega^{2k}}\mathrm{d}\zeta\right.\\
&\left.-\frac{1}{\pi}\int_{\Omega}\frac{z\overline{w_{\bar{\zeta}}(\zeta)}}{z\overline{\zeta}-\omega^{2k}}\mathrm{d}\xi\mathrm{d}\eta\right\},\quad z\in\Omega.
\end{aligned}
\tag{2.20}
$$

Subtracting the sum of (2.19) and (2.20) from (2.11), we easily get

$$
w(z) = I_1 + I_2 + I_3,\quad z\in\Omega \tag{2.21}
$$

with

$$
\begin{aligned}
I_1 = \frac{1}{2\pi i}\int_{L}\sum_{k=0}^{n-1}&\left[w(\zeta)\left(\frac{\zeta}{\zeta-z\omega^{2k}}+\frac{z\zeta}{z\zeta-\omega^{2k}}\right)\right.\\
&\left.-\overline{w(\zeta)}\left(\frac{\overline{\zeta}}{\overline{\zeta}-z\omega^{2k}}+\frac{z\overline{\zeta}}{z\overline{\zeta}-\omega^{2k}}\right)\right]\frac{\mathrm{d}\zeta}{\zeta},
\end{aligned}
\tag{2.22}
$$

$$
\begin{aligned}
I_2 = \frac{1}{2\pi i}\int_{[\omega,0]\cup[0,1]}\sum_{k=0}^{n-1}&\left[w(\zeta)\left(\frac{1}{\zeta-z\omega^{2k}}+\frac{z}{z\zeta-\omega^{2k}}\right)\right.\\
&\left.+\overline{w(\zeta)}\left(\frac{1}{\zeta-z\omega^{2k}}+\frac{z}{z\zeta-\omega^{2k}}\right)\right]\mathrm{d}\zeta
\end{aligned}
\tag{2.23}
$$

and

$$I_3 = -\frac{1}{\pi}\int\limits_{\Omega}\sum_{k=0}^{n-1}\left[w_{\bar\zeta}(\zeta)\left(\frac{1}{\zeta - z\omega^{2k}} - \frac{z}{\omega^{2k} - z\zeta}\right)\right. \\ \left. -\overline{w_{\bar\zeta}(\zeta)}\left(\frac{1}{\overline{\zeta} - z\omega^{2k}} - \frac{z}{\omega^{2k} - z\overline{\zeta}}\right)\right]\mathrm{d}\xi\mathrm{d}\eta. \tag{2.24}$$

By simple computation, we have

$$I_1 = \frac{1}{2\pi i}\int\limits_{L}\mathrm{Re}w(\zeta)\sum_{k=0}^{n-1}\left(\frac{\zeta + z\omega^{2k}}{\zeta - z\omega^{2k}} - \frac{\overline{\zeta} + z\omega^{-2k}}{\overline{\zeta} - z\omega^{-2k}}\right)\frac{\mathrm{d}\zeta}{\zeta} + \frac{n}{\pi}\int\limits_{L}\frac{\mathrm{Im}w(\zeta)}{\zeta}\mathrm{d}\zeta \tag{2.25}$$

and

$$I_2 = \frac{1}{\pi i}\int\limits_{[\omega,0]\cup[0,1]}\mathrm{Re}w(\zeta)\sum_{k=0}^{n-1}\left(\frac{1}{\zeta - z\omega^{2k}} - \frac{z}{\omega^{2k} - z\zeta}\right)\mathrm{d}\zeta. \tag{2.26}$$

This completes the proof.

**Remark 2.2.1.** When $n = 1$, the sector $\Omega$ is the upper half disc and Theorem 2.2.1 coincides with Theorem 1 in [26].

### 2.2.2 Schwarz Problem

Firstly, the boundary behavior of some linear integrals are investigated. Let

$$K(z,\zeta) = \sum_{k=0}^{n-1}\left(\frac{1}{\zeta - z\omega^{2k}} - \frac{1}{\zeta - \overline{z}\omega^{-2k}} + \frac{z}{z\zeta - \omega^{-2k}} - \frac{\overline{z}}{\overline{z}\zeta - \omega^{2k}}\right). \tag{2.27}$$

**Lemma 2.2.2.** *If $\gamma \in C(L;\mathbb{C})$, then*

$$\begin{cases} \lim\limits_{z\in\Omega,\ z\to t}\dfrac{1}{2\pi i}\displaystyle\int\limits_{L}\left[\gamma(\zeta) - \gamma(1)\right]K(z,\zeta)\mathrm{d}\zeta = \gamma(t) - \gamma(1), & t \in L\setminus\{\omega\}, \\ \lim\limits_{z\in\Omega,\ z\to t}\dfrac{1}{2\pi i}\displaystyle\int\limits_{L}\left[\gamma(\zeta) - \gamma(\omega)\right]K(z,\zeta)\mathrm{d}\zeta = \gamma(t) - \gamma(\omega), & t \in L\setminus\{1\} \end{cases} \tag{2.28}$$

*and*

$$\lim_{z\in\Omega,\ z\to t}\frac{1}{2\pi i}\int\limits_{L}\gamma(\zeta)K(z,\zeta)\mathrm{d}\zeta = 0, \quad t \in (\omega,0]\cup[0,1), \tag{2.29}$$

*where $K$ is defined by (2.27).*

**Proof:** Simple computation gives

$$\lim_{z\in\Omega,\ z\to t}\sum_{k=1}^{n-1}\left(\frac{1}{\zeta - z\omega^{2k}} - \frac{1}{\zeta - \overline{z}\omega^{-2k}} + \frac{z}{z\zeta - \omega^{2k}} - \frac{\overline{z}}{\overline{z}\zeta - \omega^{-2k}}\right) = 0$$

for $t \in L\setminus\{\omega\}$ and $\zeta \in L$ by Lemma 2.2.1. Hence we see that

$$\begin{aligned}
&\lim_{z\in\Omega,\ z\to t}\frac{1}{2\pi i}\int_L [\gamma(\zeta)-\gamma(1)]\,K(z,\zeta)\mathrm{d}\zeta \\
&= \lim_{z\in\Omega,\ z\to t}\frac{1}{2\pi i}\int_L [\gamma(\zeta)-\gamma(1)]\left(\frac{\zeta}{\zeta - z} - \frac{\overline{\zeta}}{\overline{\zeta} - z} + \frac{\overline{\zeta}}{\overline{\zeta} - \overline{z}} - \frac{\zeta}{\zeta - \overline{z}}\right)\frac{\mathrm{d}\zeta}{\zeta} \\
&= \lim_{z\in\Omega,\ z\to t}\frac{1}{2\pi i}\int_{L\cup\overline{L}} \Lambda_1(\zeta)\left(\frac{\zeta}{\zeta - z} + \frac{\overline{\zeta}}{\overline{\zeta} - \overline{z}} - 1\right)\frac{\mathrm{d}\zeta}{\zeta},\ \ t\in L\setminus\{\omega\},
\end{aligned} \tag{2.30}$$

where $\overline{L} = \{\tau:\ \tau = e^{i\varphi},\ -\frac{\pi}{n} < \varphi < 0\}$ is oriented counter-clockwise and

$$\Lambda_1(\zeta) = \begin{cases} \gamma(\zeta)-\gamma(1), & \zeta\in L, \\ -\gamma(\overline{\zeta})+\gamma(1), & \zeta \in \overline{L}. \end{cases}$$

Therefore, by the continuity of $\Lambda_1(\zeta)$ on $L\cup\overline{L}$ and the boundary property of the classical Poisson kernel on the unit circle, (2.30) implies that

$$\lim_{z\in\Omega,\ z\to t}\frac{1}{2\pi i}\int_L [\gamma(\zeta)-\gamma(1)]\,K(z,\zeta)\mathrm{d}\zeta = \gamma(t)-\gamma(1),\ \ t\in L\setminus\{\omega\}.$$

Similarly, the other equality in (2.28) is valid. Finally, if $\zeta \in L$ and $z \in (\omega, 0]\cup[0,1)$, then $K(z,\zeta)\equiv 0$, and hence (2.29) holds.

**Lemma 2.2.3.** *If* $\gamma \in C([0,1];\mathbb{C})$*, then*

$$\lim_{z\in\Omega,\ z\to t}\frac{1}{2\pi i}\int_0^1 [\gamma(\zeta)-\gamma(1)]\,K(z,\zeta)\mathrm{d}\zeta = \gamma(t)-\gamma(1),\ \ t\in(0,1], \tag{2.31}$$

$$\lim_{z\in\Omega,\ z\to t}\frac{1}{2\pi i}\int_0^1 \gamma(\zeta)K(z,\zeta)\mathrm{d}\zeta = 0,\ \ t\in\partial\Omega\setminus[0,1], \tag{2.32}$$

*where* $K$ *is given by* (2.27).

**Proof:** Similarly as before, we have

$$\begin{aligned}
&\lim_{z\in\Omega,\ z\to t}\frac{1}{2\pi i}\int\limits_0^1\left[\gamma(\zeta)-\gamma(1)\right]K(z,\zeta)\mathrm{d}\zeta\\
&=\lim_{z\in\Omega,\ z\to t}\frac{1}{2\pi i}\int\limits_0^1\left[\gamma(\zeta)-\gamma(1)\right]\left(\frac{1}{\zeta-z}-\frac{1}{\zeta-\overline{z}}-\frac{\overline{z}}{\overline{z}\zeta-1}+\frac{z}{z\zeta-1}\right)\mathrm{d}\zeta\\
&=\lim_{z\in\Omega,\ z\to t}\frac{1}{2\pi i}\int\limits_0^{+\infty}\Lambda_2(\zeta)\left(\frac{1}{\zeta-z}-\frac{1}{\zeta-\overline{z}}\right)\mathrm{d}\zeta,\ \ t\in(0,1]
\end{aligned}$$

with

$$\Lambda_2(\zeta)=\begin{cases}\gamma(\zeta)-\gamma(1), & \zeta\in(0,1],\\ -\gamma(\frac{1}{\zeta})+\gamma(1), & \zeta\in(1,+\infty).\end{cases}$$

Thus, from the continuity of $\Lambda_2$ on $(0,+\infty)$ and the property of the Poisson kernel on the real axis, (2.31) is obtained. If $\zeta\in[0,1]$ and $z\in\partial\Omega\setminus[0,1]$, then $K(z,\zeta)\equiv 0$, and hence (2.32) is true.

Similarly to Lemma 2.2.3, the following lemma is valid.

**Lemma 2.2.4.** *If* $\gamma\in C([\omega,0];\mathbb{C})$*, then*

$$\lim_{z\in\Omega,\ z\to t}\frac{1}{2\pi i}\int\limits_{[\omega,0]}\left[\gamma(\zeta)-\gamma(\omega)\right]K(z,\zeta)\mathrm{d}\zeta=\gamma(t)-\gamma(\omega),\ \ t\in[\omega,0),\tag{2.33}$$

$$\lim_{z\in\Omega,\ z\to t}\frac{1}{2\pi i}\int\limits_{[\omega,0]}\gamma(\zeta)K(z,\zeta)\mathrm{d}\zeta=0,\ \ t\in\partial\Omega\setminus[\omega,0],\tag{2.34}$$

*where* $K$ *is given by* (2.27)*.*

It should be noted that the boundary behavior at the corner $z=0$ needs to be especially investigated because 0 is the common point of the boundary of all the sectors $\Omega_k, E_k$, $k=0,1,\cdots,n-1$.

**Lemma 2.2.5.** *If* $\gamma\in C([\omega,0]\cup[0,1];\mathbb{C})$*, then*

$$\lim_{z\in\Omega,\ z\to 0}\frac{1}{2\pi i}\int\limits_{[\omega,0]\cup[0,1]}\left[\gamma(\zeta)-\gamma(0)\right]K(z,\zeta)\mathrm{d}\zeta=0,\tag{2.35}$$

*where* $K$ *is defined by* (2.27)*.*

**Proof:** Firstly, we have

$$\begin{aligned}
&\lim_{z\in\Omega,\ z\to 0}\frac{1}{2\pi i}\int\limits_{[\omega,0]\cup[0,1]}[\gamma(\zeta)-\gamma(0)]\,K(z,\zeta)\mathrm{d}\zeta\\
&=\lim_{z\in\Omega,\ z\to 0}\frac{1}{2\pi i}\int\limits_{[\omega,0]\cup[0,1]}[\gamma(\zeta)-\gamma(0)]\sum_{k=0}^{n-1}\left(\frac{1}{\zeta-z\omega^{2k}}-\frac{1}{\zeta-\bar{z}\omega^{-2k}}\right)\mathrm{d}\zeta.
\end{aligned}\tag{2.36}$$

Next we discuss the boundary behavior at $z=0$ in two cases.

Case 1: if $n$ is an even number, by (2.36),

$$\begin{aligned}
&\lim_{z\in\Omega,\ z\to 0}\frac{1}{2\pi i}\int_0^1[\gamma(\zeta)-\gamma(0)]\,K(z,\zeta)\mathrm{d}\zeta\\
&=\lim_{z\in\Omega,\ z\to 0}\frac{1}{2\pi i}\int_0^1[\gamma(\zeta)-\gamma(0)]\sum_{k=0}^{\frac{n}{2}-1}\left(\frac{1}{\zeta-z\omega^{2k}}-\frac{1}{\zeta-\bar{z}\omega^{-2k}}\right.\\
&\qquad\qquad\left.+\frac{1}{\zeta+z\omega^{2k}}-\frac{1}{\zeta+\bar{z}\omega^{-2k}}\right)\mathrm{d}\zeta\\
&=\sum_{k=0}^{\frac{n}{2}-1}\lim_{z\in\Omega,\ z\to 0}\frac{1}{2\pi i}\int_{-1}^1\Lambda_3(\zeta)\left(\frac{1}{\zeta-z\omega^{2k}}-\frac{1}{\zeta-\bar{z}\omega^{-2k}}\right)\mathrm{d}\zeta
\end{aligned}\tag{2.37}$$

with

$$\Lambda_3(\zeta)=\begin{cases}\gamma(\zeta)-\gamma(0), & \zeta\in(0,1],\\ -\gamma(-\zeta)+\gamma(0), & \zeta\in[-1,0).\end{cases}$$

Since for $z\in\Omega$, all $z\omega^{2k}$, $k=0,\cdots,\frac{n}{2}-1$ are in the upper half-plane, (2.37) leads to

$$\lim_{z\in\Omega,\ z\to 0}\frac{1}{2\pi i}\int_0^1[\gamma(\zeta)-\gamma(0)]\,K(z,\zeta)\mathrm{d}\zeta=\frac{n}{2}\Lambda_3(0)=0.\tag{2.38}$$

Similarly,

$$\lim_{z\in\Omega,\ z\to 0}\frac{1}{2\pi i}\int\limits_{[\omega,0]}[\gamma(\zeta)-\gamma(0)]\,K(z,\zeta)\mathrm{d}\zeta=0.\tag{2.39}$$

Thus, the sum of (2.38) and (2.39) gives the desired conclusion (2.35).

Case 2: if $n$ is an odd number, then (2.36) equals

$$\begin{aligned}
&\lim_{z\in\Omega,\ z\to 0}\Bigg\{\frac{1}{2\pi i}\int_0^1[\gamma(\zeta)-\gamma(0)]\sum_{k=0}^{\frac{n-1}{2}}\left(\frac{1}{\zeta-z\omega^{2k}}-\frac{1}{\zeta-\bar{z}\omega^{-2k}}\right)\mathrm{d}\zeta\\
&+\frac{1}{2\pi i}\int_0^1[\gamma(\zeta)-\gamma(0)]\sum_{k=1}^{\frac{n-1}{2}}\left(\frac{1}{\zeta+z\omega^{2k-1}}-\frac{1}{\zeta+\bar{z}\omega^{-(2k-1)}}\right)\mathrm{d}\zeta\\
&-\frac{1}{2\pi i}\int_0^1[\gamma(\zeta\omega)-\gamma(0)]\sum_{k=1}^{\frac{n-1}{2}}\left(\frac{1}{\zeta-z\omega^{2k-1}}-\frac{1}{\zeta-\bar{z}\omega^{-(2k-1)}}\right)\mathrm{d}\zeta\\
&-\frac{1}{2\pi i}\int_0^1[\gamma(\zeta\omega)-\gamma(0)]\sum_{k=0}^{\frac{n-1}{2}}\left(\frac{1}{\zeta+z\omega^{2k}}-\frac{1}{\zeta+\bar{z}\omega^{-2k}}\right)\mathrm{d}\zeta\Bigg\}\\
=&\lim_{z\in\Omega,\ z\to 0}\sum_{k=0}^{\frac{n-1}{2}}\frac{1}{2\pi i}\int_{-1}^1\Lambda_4(\zeta)\left(\frac{1}{\zeta-z\omega^{2k}}-\frac{1}{\zeta-\bar{z}\omega^{-2k}}\right)\mathrm{d}\zeta\\
&-\lim_{z\in\Omega,\ z\to 0}\sum_{k=1}^{\frac{n-1}{2}}\frac{1}{2\pi i}\int_{-1}^1\Lambda_5(\zeta)\left(\frac{1}{\zeta-z\omega^{2k-1}}-\frac{1}{\zeta-\bar{z}\omega^{-(2k-1)}}\right)\mathrm{d}\zeta
\end{aligned}\tag{2.40}$$

with

$$\Lambda_4(\zeta)=\begin{cases}\gamma(\zeta)-\gamma(0), & \zeta\in[0,1),\\ \gamma(-\zeta\omega)-\gamma(0), & \zeta\in[-1,0),\end{cases}$$

and

$$\Lambda_5(\zeta)=\begin{cases}\gamma(\zeta\omega)-\gamma(0), & \zeta\in[0,1),\\ \gamma(-\zeta)-\gamma(0), & \zeta\in[-1,0).\end{cases}$$

Then the continuity of $\Lambda_4,\Lambda_5$ at $\zeta=0$ implies that the limit in (2.40) equals

$$\frac{n+1}{2}\Lambda_4(0)-\frac{n-1}{2}\Lambda_5(0)=0.$$

Therefore (2.35) is valid in this case. In conclusion, the desired conclusion (2.35) is always true.

**Lemma 2.2.6.** *If* $\gamma\in C([\omega,0]\cup[0,1];\mathbb{C})$*, then*

$$\lim_{z\in\Omega,\ z\to t}\frac{1}{2\pi i}\int_{[\omega,0]\cup[0,1]}[\gamma(\zeta)-\gamma(0)]\,K(z,\zeta)\mathrm{d}\zeta=\gamma(t)-\gamma(0),\ \ t\in(\omega,0]\cup[0,1),$$

*where* $K$ *is defined by* (2.27)*.*

**Proof:** By Lemma 2.2.5, we only need to prove

$$\lim_{z\in\Omega,\ z\to t} \frac{1}{2\pi i} \int\limits_{[\omega,0]\cup[0,1]} [\gamma(\zeta)-\gamma(0)]\, K(z,\zeta)\mathrm{d}\zeta = \gamma(t)-\gamma(0),\ \ t\in(\omega,0)\cup(0,1). \tag{2.41}$$

By Lemmas 2.2.3 and 2.2.4, (2.41) is equivalent to

$$\lim_{z\in\Omega,\ z\to t} \frac{1}{2\pi i} \int\limits_{[\omega,0]} [\gamma(\zeta)-\gamma(0)]\, K(z,\zeta)\mathrm{d}\zeta = \gamma(t)-\gamma(0),\ \ t\in(\omega,0) \tag{2.42}$$

and

$$\lim_{z\in\Omega,\ z\to t} \frac{1}{2\pi i} \int_0^1 [\gamma(\zeta)-\gamma(0)]\, K(z,\zeta)\mathrm{d}\zeta = \gamma(t)-\gamma(0),\ \ t\in(0,1). \tag{2.43}$$

The left-hand side of (2.42) equals

$$\begin{aligned}
&\lim_{z\in\Omega,\ z\to t} \frac{1}{2\pi i} \int\limits_{[\omega,0]} [\gamma(\zeta)-\gamma(0)] \left(\frac{1}{\zeta-z}-\frac{1}{\zeta-\overline{z}\omega^2}\right)\mathrm{d}\zeta \\
&= -\lim_{z\in\Omega,\ z\to t} \frac{1}{2\pi i} \int_0^1 [\gamma(\zeta\omega)-\gamma(0)] \left(\frac{1}{\zeta-z\omega^{-1}}-\frac{1}{\zeta-\overline{z}\omega}\right)\mathrm{d}\zeta \\
&= \gamma(t)-\gamma(0),\ \ t\in(\omega,0).
\end{aligned}$$

Hence (2.42) is true. Similarly, (2.43) is also valid.

**Theorem 2.2.2.** *If* $\gamma\in C(\partial\Omega;\mathbb{C})$*, then*

$$\lim_{z\in\Omega,\ z\to t} \frac{1}{2\pi i} \int\limits_{\partial\Omega} \gamma(\zeta)K(z,\zeta)\mathrm{d}\zeta = \gamma(t),\ \ t\in\partial\Omega, \tag{2.44}$$

*where* $K$ *is defined by* (2.27).

**Proof:** Let $t\in(\omega,0]\cup[0,1)$, by Lemmas 2.2.2 and 2.2.6, the left-hand side in (2.44) equals

$$[\gamma(t)-\gamma(0)] + \lim_{z\in\Omega,\ z\to t} \frac{\gamma(0)}{2\pi i} \int\limits_{[\omega,0]\cup[0,1]} K(z,\zeta)\mathrm{d}\zeta. \tag{2.45}$$

On the other hand, by (2.12),

$$
\begin{aligned}
1 = \frac{1}{2\pi i}\int\limits_{L}\sum_{k=0}^{n-1}\left(\frac{\zeta + z\omega^{2k}}{\zeta - z\omega^{2k}} - \frac{\overline{\zeta} + z\omega^{-2k}}{\overline{\zeta} - z\omega^{-2k}}\right)\frac{\mathrm{d}\zeta}{\zeta} \\
+\frac{1}{\pi i}\int\limits_{[\omega,0]\cup[0,1]}\sum_{k=0}^{n-1}\left(\frac{1}{\zeta - z\omega^{2k}} - \frac{z}{\omega^{2k} - z\zeta}\right)\mathrm{d}\zeta.
\end{aligned} \tag{2.46}
$$

Taking the real part on both sides of (2.46) gives

$$
1 = \frac{1}{2\pi i}\int\limits_{L}K(z,\zeta)\mathrm{d}\zeta + \frac{1}{2\pi i}\int\limits_{[\omega,0]\cup[0,1]}K(z,\zeta)\mathrm{d}\zeta. \tag{2.47}
$$

Then, by (2.47) and (2.29), (2.45) equals

$$
\gamma(t) - \lim_{z\in\Omega,\ z\to t}\frac{\gamma(0)}{2\pi i}\int\limits_{L}K(z,\zeta)\mathrm{d}\zeta = \gamma(t).
$$

Therefore, (2.44) is valid for $t \in (\omega, 0] \cup [0, 1)$. Similarly, (2.44) is also true for $t \in \partial\Omega \setminus [0, 1]$ and $t \in \partial\Omega \setminus [\omega, 0]$, respectively. This completes the proof.

We introduce the Schwarz-type operator as follows

$$
\begin{aligned}
S[\gamma](z) = \ & \frac{1}{2\pi i}\int\limits_{L}\gamma(\zeta)\sum_{k=0}^{n-1}\left(\frac{\zeta + z\omega^{2k}}{\zeta - z\omega^{2k}} - \frac{\overline{\zeta} + z\omega^{-2k}}{\overline{\zeta} - z\omega^{-2k}}\right)\frac{\mathrm{d}\zeta}{\zeta} \\
& +\frac{1}{\pi i}\int\limits_{[\omega,0]\cup[0,1]}\gamma(\zeta)\sum_{k=0}^{n-1}\left(\frac{1}{\zeta - z\omega^{2k}} - \frac{z}{\omega^{2k} - z\zeta}\right)\mathrm{d}\zeta,\ \ z \in \Omega,
\end{aligned} \tag{2.48}
$$

where $\gamma \in C(\partial\Omega; \mathbb{R})$. Obviously, $S[\gamma](z)$ is analytic in the domain $\Omega$, denoted as $S[\gamma] \in A(\Omega)$. Further,

$$
\mathrm{Re}S[\gamma](z) = \frac{1}{2\pi i}\int\limits_{\partial\Omega}\gamma(\zeta)K(z,\zeta)\mathrm{d}\zeta,\ \ z \in \Omega \tag{2.49}
$$

for $\gamma \in C(\partial\Omega; \mathbb{R})$, where $K$ is defined by (2.27). Thus, by Theorem 2.2.2, the following result is valid.

**Theorem 2.2.3.** If $\gamma \in C(\partial\Omega; \mathbb{R})$, then $\{\mathrm{Re}S[\gamma]\}^{+}(t) = \gamma(t),\ \ t \in \partial\Omega$, where $S$ is the Schwarz-type operator defined by (2.48).

Finally, a Pompeiu-type operator for $\Omega$ is introduced by

$$\begin{aligned}\widehat{T}[f](z) = -\frac{1}{\pi}\int_{\Omega}\sum_{k=0}^{n-1}\Big[&f(\zeta)\Big(\frac{1}{\zeta - z\omega^{2k}} - \frac{z}{\omega^{2k} - z\zeta}\Big)\\ &-\overline{f(\zeta)}\Big(\frac{1}{\overline{\zeta} - z\omega^{2k}} - \frac{z}{\omega^{2k} - z\overline{\zeta}}\Big)\Big]\mathrm{d}\xi\mathrm{d}\eta, \quad z \in \Omega,\end{aligned} \tag{2.50}$$

where $f \in L_p(\Omega;\mathbb{C})$, $p > 2$. By simple computation, we have

$$\mathrm{Re}\widehat{T}[f](z) = -\frac{1}{2\pi}\int_{\Omega}\Big[f(\zeta)K(z,\zeta) - \overline{f(\zeta)}K(z,\overline{\zeta})\Big]\mathrm{d}\xi\mathrm{d}\eta, \quad z \in \Omega, \tag{2.51}$$

where $K$ is defined by (2.27).

**Theorem 2.2.4.** *If $f \in L_p(\Omega;\mathbb{C})$, $p > 2$, then $\partial_{\overline{z}}\widehat{T}[f](z) = f(z)$, $z \in \Omega$ in weak sense, and $\{\mathrm{Re}\widehat{T}[f]\}^+(t) = 0,\ t \in \partial\Omega$, where $\widehat{T}$ is the Pompeiu-type operator defined by* (2.50)*.*

**Proof:** By (2.50), it is obvious that $\partial_{\overline{z}}\widehat{T}[f](z) = \partial_{\overline{z}}T[f](z) = f(z)$, $z \in \Omega$ in weak sense, where $T$ is defined by (2.5). On the other hand, $K(z,\zeta) = K(z,\overline{\zeta}) = 0$, $(z,\zeta) \in \partial\Omega \times \Omega$ implies that $\{\mathrm{Re}\widehat{T}[f]\}^+(t) = 0,\ t \in \partial\Omega$.

Therefore, we obtain the following Schwarz problem.

**Theorem 2.2.5.** *The Schwarz problem*

$$\begin{cases} \partial_{\overline{z}}w = f \ \text{ in } \Omega, \quad \mathrm{Re}w = \gamma \ \text{ on } \partial\Omega, \\ \displaystyle\frac{n}{\pi}\int_0^{\frac{\pi}{n}} \mathrm{Im}w(e^{i\varphi})\mathrm{d}\varphi = c, \quad c \in \mathbb{R} \end{cases} \tag{2.52}$$

*for $f \in L_p(\Omega;\mathbb{C})$, $p > 2$, $\gamma \in C(\partial\Omega;\mathbb{R})$, is uniquely solvable by*

$$w(z) = S[\gamma](z) + \widehat{T}[f](z) + ic, \tag{2.53}$$

*where $S, \widehat{T}$ are defined by* (2.48) *and* (2.50)*, respectively.*

**Proof:** By Theorem 2.2.1, if there is a solution for the Schwarz problem (2.52), it must be of the form (2.53). Thus we only need to verify that (2.53) provides a solution.

If $w$ is given by (2.53), then $\partial_{\overline{z}} w = f$ in $\Omega$ since $S[\gamma] \in A(\Omega)$ and $\partial_{\overline{z}} \widehat{T}[f] = f$ by Theorem 2.2.4. Also by Theorems 2.2.3 and 2.2.4, $\mathrm{Re} w = \gamma$ on $\partial\Omega$. Finally, we only need to prove that

$$\frac{n}{\pi} \int_0^{\frac{\pi}{n}} \mathrm{Im} w(e^{i\varphi}) \mathrm{d}\varphi = c, \quad c \in \mathbb{R}. \tag{2.54}$$

Actually,

$$\frac{1}{2\pi i} \int_L \sum_{k=0}^{n-1} \left( \frac{\zeta + z\omega^{2k}}{\zeta - z\omega^{2k}} - \frac{\overline{\zeta} + z\omega^{-2k}}{\overline{\zeta} - z\omega^{-2k}} \right) \frac{\mathrm{d}z}{z} = \frac{1}{2\pi i} \int_L \sum_{k=0}^{n-1} \left( \frac{\zeta + z\omega^{2k}}{\zeta - z\omega^{2k}} + \frac{\zeta + \overline{z}\omega^{2k}}{\zeta - \overline{z}\omega^{2k}} \right) \frac{\mathrm{d}z}{z}$$
$$= \frac{1}{2\pi i} \int_{|z|=1} \frac{\zeta + z}{\zeta - z} \frac{\mathrm{d}z}{z} = \frac{1}{2\pi i} \int_{|z|=1} \left( \frac{1}{z} - \frac{2}{z - \zeta} \right) \mathrm{d}z = 0, \ \ \zeta \in L$$

and

$$\frac{1}{2\pi i} \int_L \sum_{k=0}^{n-1} \left( \frac{1}{\zeta - z\omega^{2k}} - \frac{z}{\omega^{2k} - z\zeta} \right) \frac{\mathrm{d}z}{z} = \frac{1}{2\pi i} \int_L \sum_{k=0}^{n-1} \left( \frac{1}{\zeta - z\omega^{2k}} + \frac{1}{\zeta - \overline{z}\omega^{2k}} \right) \frac{\mathrm{d}z}{z}$$
$$= \frac{1}{2\pi i} \int_{|z|=1} \frac{1}{\zeta - z} \frac{\mathrm{d}z}{z} = 0 \ \text{ for } \ \ \zeta \in (\omega, 0] \cup [0, 1).$$

Hence

$$\int_L \mathrm{Im} S[\gamma](z) \frac{\mathrm{d}z}{z} = 0. \tag{2.55}$$

Similarly, for $\zeta \in \Omega$,

$$\frac{1}{2\pi i} \int_L \sum_{k=0}^{n-1} \left( \frac{1}{\zeta - z\omega^{2k}} - \frac{z}{\omega^{2k} - z\zeta} \right) \frac{\mathrm{d}z}{z}$$
$$= \frac{1}{2\pi i} \int_L \sum_{k=0}^{n-1} \left( \frac{1}{\overline{\zeta} - z\omega^{2k}} - \frac{z}{\omega^{2k} - z\overline{\zeta}} \right) \frac{\mathrm{d}z}{z} = 0,$$

which implies that

$$\int_L \mathrm{Im} \widehat{T}[f](z) \frac{\mathrm{d}z}{z} = 0. \tag{2.56}$$

Thus, (2.53), (2.55) and (2.56) lead to (2.54). Then the proof is completed.

## 2.3 Dirichlet Problem with Angle $\pi/n$ ($n \in \mathbb{N}$)

We firstly consider the Dirichlet problem for the homogeneous Cauchy-Riemann equation

$$\begin{cases} \partial_{\bar{z}} w(z) = 0, & z \in \Omega, \\ w^+(t) = \gamma(t), & t \in \partial\Omega, \end{cases} \tag{2.57}$$

where the boundary data $\gamma \in C(\partial\Omega; \mathbb{C})$.

**Theorem 2.3.1.** *The Dirichlet problem* (2.57) *is solvable if and only if*

$$\frac{1}{2\pi i} \int_{\partial\Omega} \gamma(\zeta) \sum_{k=0}^{n-1} \left( \frac{1}{\zeta - \bar{z}\omega^{2k}} - \frac{\bar{z}}{\omega^{2k} - \bar{z}\zeta} \right) \mathrm{d}\zeta = 0, \quad z \in \Omega, \tag{2.58}$$

*and its solution is uniquely expressed as*

$$w(z) = \frac{1}{2\pi i} \int_{\partial\Omega} \gamma(\zeta) \sum_{k=0}^{n-1} \left( \frac{1}{\zeta - z\omega^{2k}} - \frac{z}{\omega^{2k} - z\zeta} \right) \mathrm{d}\zeta, \quad z \in \Omega. \tag{2.59}$$

**Proof:** By (2.11) in Theorem 2.2.1, if there is a solution for the Dirichlet problem (2.57), then its solution can be represented as (2.59).

Now we prove that (2.58) is a necessary condition. If $w$ is the solution to the Dirichlet problem (2.57) given by (2.59), then

$$w^+(t) = \gamma(t), \qquad t \in \partial\Omega. \tag{2.60}$$

Let

$$h(z) = \frac{1}{2\pi i} \int_{\partial\Omega} \gamma(\zeta) \sum_{k=0}^{n-1} \left( \frac{1}{\zeta - \bar{z}\omega^{2k}} - \frac{\bar{z}}{\omega^{2k} - \bar{z}\zeta} \right) \mathrm{d}\zeta, \quad z \in \Omega.$$

Obviously,

$$w(z) - h(z) = \frac{1}{2\pi i} \int_{\partial\Omega} \gamma(\zeta) K(z, \zeta) \mathrm{d}\zeta,$$

where $K$ is defined by (2.27). By Theorem 2.2.2,

$$\lim_{z \in \Omega,\ z \to t} [w(z) - h(z)] = \gamma(t), \quad t \in \partial\Omega. \tag{2.61}$$

By (2.60) and (2.61), we have $h^+(t) = 0$, $t \in \partial\Omega$. Since $\overline{h(z)}$ is analytic for $z \in \Omega$, then by the maximum principle of analytic functions, $h(z) \equiv 0$ for $z \in \Omega$, which is just the condition (2.58).

On the other hand, If the condition (2.58) is satisfied, then

$$w(z) = \frac{1}{2\pi i}\int\limits_{\partial\Omega} \gamma(\zeta)\sum_{k=0}^{n-1}\left(\frac{1}{\zeta - z\omega^{2k}} + \frac{z}{z\zeta - \omega^{2k}}\right)d\zeta = \frac{1}{2\pi i}\int\limits_{\partial\Omega}\gamma(\zeta)K(z,\zeta)d\zeta, \quad z \in \Omega.$$

Hence, obviously $\partial_{\bar{z}}w(z) = 0$, $z \in \Omega$, and $w^+(t) = \gamma(t)$, $t \in \partial\Omega$ by Theorem 2.2.2.

**Theorem 2.3.2.** *The inhomogeneous Dirichlet problem*

$$\begin{cases} \partial_{\bar{z}}w(z) = f(z), & z \in \Omega, \quad f \in L_p(\Omega;\mathbb{C}),\ p > 2, \\ w^+(t) = \gamma(t), & t \in \partial\Omega, \quad \gamma \in C(\partial\Omega;\mathbb{C}) \end{cases} \tag{2.62}$$

*is solvable if and only if for $z \in \Omega$,*

$$\begin{aligned} &\frac{1}{2\pi i}\int\limits_{\partial\Omega}\gamma(\zeta)\sum_{k=0}^{n-1}\left(\frac{1}{\zeta - \bar{z}\omega^{2k}} + \frac{\bar{z}}{\bar{z}\zeta - \omega^{2k}}\right)d\zeta \\ &= \frac{1}{\pi}\int\limits_{\Omega} f(\zeta)\sum_{k=0}^{n-1}\left(\frac{1}{\zeta - \bar{z}\omega^{2k}} + \frac{\bar{z}}{\bar{z}\zeta - \omega^{2k}}\right)d\xi d\eta, \end{aligned} \tag{2.63}$$

*and its solution can be uniquely expressed as*

$$\begin{aligned} w(z) &= \frac{1}{2\pi i}\int\limits_{\partial\Omega}\gamma(\zeta)\sum_{k=0}^{n-1}\left(\frac{1}{\zeta - z\omega^{2k}} + \frac{z}{z\zeta - \omega^{2k}}\right)d\zeta \\ &\quad -\frac{1}{\pi}\int\limits_{\Omega} f(\zeta)\sum_{k=0}^{n-1}\left(\frac{1}{\zeta - z\omega^{2k}} + \frac{z}{z\zeta - \omega^{2k}}\right)d\xi d\eta, \quad z \in \Omega. \end{aligned} \tag{2.64}$$

**Proof:** If the Dirichlet problem (2.62) is solvable, then its solution can be represented as (2.64) due to Theorem 2.2.1. Let $\phi(z) = w(z) - T[f](z)$, then

$$\partial_{\bar{z}}\phi = 0 \text{ in } \Omega, \quad \phi = \gamma - T[f] \text{ on } \partial\Omega,$$

where $T$ is defined by (2.5). From Theorem 2.3.1, the condition of solvability is

$$\frac{1}{2\pi i}\int\limits_{\partial\Omega}\{\gamma(\zeta)-T[f](\zeta)\}\sum_{k=0}^{n-1}\left(\frac{1}{\zeta-\bar{z}\omega^{2k}}-\frac{\bar{z}}{\omega^{2k}-\bar{z}\zeta}\right)\mathrm{d}\zeta=0,$$

which is just condition (2.63) by computation.

Conversely, if the condition of solvability (2.63) is satisfied, (2.64) can be rewritten as

$$w(z)=\frac{1}{2\pi i}\int\limits_{\partial\Omega}\gamma(\zeta)K(z,\zeta)\mathrm{d}\zeta-\frac{1}{\pi}\int\limits_{\Omega}f(\zeta)K(z,\zeta)\mathrm{d}\xi\mathrm{d}\eta,\ \ z\in\Omega. \tag{2.65}$$

Since $-\frac{1}{\pi}\int\limits_{\Omega}f(\zeta)K(z,\zeta)\mathrm{d}\xi\mathrm{d}\eta$ tends to 0 as $z\to t\in\partial\Omega$, (2.65) implies that $w^+(t)=\gamma(t)$, $t\in\partial D$ by Theorem 2.2.2. Obviously, we also have $\partial_{\bar{z}}w(z)=f(z)$, $z\in\Omega$. This completes the proof.

## 2.4 Schwarz Problem with Angle $\pi/\alpha$ $(\alpha\geq 1/2)$

In Section 2.2, we give the Schwarz-Poisson formula for the domain $\Omega$ with angle $\pi/n$ $(n\in\mathbb{N})$ explicitly by the reflection method. However, for a general angle $\pi/\alpha$ $(\alpha\geq 1/2)$, it is difficult to bring this formula into effect likewise. This Section is devoted to extending the Schwarz-Poisson representation formula to a fan-shaped domain $\Omega^+$ with angle $\pi/\alpha$ by a proper conformal mapping and to solving the Schwarz problem in detail.

Let $\Omega^+$ be a fan-shaped domain with angle $\vartheta=\frac{\pi}{\alpha}$ $(\alpha\geq 1/2)$, that is,

$$\Omega^+=\left\{|z|<1,\ 0<\arg z<\frac{\pi}{\alpha}\right\}.$$

Its boundary $\partial\Omega^+=[0,1]\cup\Gamma_0\cup[\varpi,0]$ is oriented counter-clockwise with $\varpi=e^{i\vartheta}$. Further, $0,\ 1,\ \varpi$ are the three corner points and the oriented segment $\Gamma_0$ is parameterized by

$$\Gamma_0:\ \tau\longmapsto e^{i\tau},\ \ \tau\in\left[0,\ \frac{\pi}{\alpha}\right].$$

Obviously, the domain $\Omega^+=\Omega$ for $\alpha=n$.

Similarly, except for additional indication, we always assume that $\alpha$ $(\alpha\geq 1/2)$ is a fixed real constant, $\vartheta=\pi/\alpha$ and $\varpi=e^{i\vartheta}$.

### 2.4.1 Schwarz-Poisson Representation

From [26] and Theorem 2.2.1, when $n = 1$, the following lemma holds.

**Lemma 2.4.1.** *Any $w \in C^1(\mathbb{D}^+;\mathbb{C}) \cap C(\overline{\mathbb{D}^+};\mathbb{C})$ can be represented as*

$$\begin{aligned}&\frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}^+} w(\zeta)\left[\frac{1}{\zeta - z} + \frac{z}{z\zeta - 1}\right]\mathrm{d}\zeta \\ &-\frac{1}{\pi}\int\limits_{\mathbb{D}^+} w_{\bar\zeta}(\zeta)\left[\frac{1}{\zeta - z} + \frac{z}{z\zeta - 1}\right]\mathrm{d}\xi\mathrm{d}\eta = \begin{cases} w(z), & z \in \mathbb{D}^+, \\ 0, & z \notin \overline{\mathbb{D}^+}, \end{cases}\end{aligned} \tag{2.66}$$

*and for $z \in \mathbb{D}^+$*

$$\begin{aligned} w(z) &= \frac{1}{2\pi i}\int\limits_{L_0} \mathrm{Re}w(\zeta)\left(\frac{\zeta + z}{\zeta - z} - \frac{\overline{\zeta} + z}{\overline{\zeta} - z}\right)\frac{\mathrm{d}\zeta}{\zeta} + \frac{1}{\pi}\int\limits_{L_0}\frac{\mathrm{Im}w(\zeta)}{\zeta}\mathrm{d}\zeta \\ &\quad + \frac{1}{\pi i}\int\limits_{-1}^{1} \mathrm{Re}w(\zeta)\left(\frac{1}{\zeta - z} - \frac{z}{1 - z\zeta}\right)\mathrm{d}\zeta \\ &\quad - \frac{1}{\pi}\int\limits_{\mathbb{D}^+}\left[w_{\bar\zeta}(\zeta)\Big(\frac{1}{\zeta - z} - \frac{z}{1 - z\zeta}\Big) - \overline{w_{\bar\zeta}(\zeta)}\Big(\frac{1}{\overline{\zeta} - z} - \frac{z}{1 - z\overline{\zeta}}\Big)\right]\mathrm{d}\xi\mathrm{d}\eta, \end{aligned} \tag{2.67}$$

*where $\mathbb{D}^+ = \{|z| < 1,\ \mathrm{Im}z > 0\}$ and $L_0 = \{|\tau| = 1,\ \mathrm{Im}\tau > 0\}$.*

We consider the conformal mapping [30]

$$\begin{aligned}\zeta :\ \ &\Omega^+ \to \mathbb{D}^+ \\ &z \mapsto z^\alpha\end{aligned} \tag{2.68}$$

with the branch cut along $(0, +\infty)$, which especially maps the boundary $\Gamma_0$ onto $L_0$, $[\varpi, 0]$ onto $[-1, 0]$ and [0,1] onto [0,1], respectively. The inverse mapping is

$$\begin{aligned}\varsigma :\ \ &\mathbb{D}^+ \to \Omega^+ \\ &z \mapsto z^{1/\alpha}\end{aligned} \tag{2.69}$$

which transforms $L_0$ onto $\Gamma_0$, $[-1, 0]$ onto $[\varpi, 0]$ and [0,1] onto [0,1], respectively.

**Remark 2.4.1.** Obviously, the above two conformal mappings can be firstly 1-1 extended to the boundary except for 0. Since the argument at 0 can be arbitrary and $\alpha \geq 1/2$, we can assume that the argument of $z^\alpha$ and $z^{1/\alpha}$ at $z = 0$ are 0. Hence, the above 1-1 mapping between the boundary is proper.

In what follows, the main analytic branches of $z^\alpha$ and $z^{1/\alpha}$ are always chosen as above, respectively. Moreover, on the basis of the analytic branches of $z^\alpha$ and $z^{1/\alpha}$, we define the analytic branch of several functions as follows,

$$z^{\alpha-1} = \frac{z^\alpha}{z}, \quad z^{1-\alpha} = \frac{z}{z^\alpha}, \quad z^{1/\alpha-1} = \frac{z^{1/\alpha}}{z}.$$

Then, based on Lemma 2.4.1 and the conformal mapping, we obtain the following Schwarz-Poisson representation for an arbitrary angle $\frac{\pi}{\alpha}$ $(\alpha \geq 1/2)$.

**Theorem 2.4.1.** *Any* $w \in C^1(\Omega^+;\mathbb{C}) \cap C(\overline{\Omega^+};\mathbb{C})$ *can be represented as*

$$\begin{aligned}&\frac{\alpha}{2\pi i}\int\limits_{\partial\Omega^+} w(\zeta)\left[\frac{1}{\zeta^\alpha - z^\alpha} + \frac{z^\alpha}{z^\alpha\zeta^\alpha - 1}\right]\zeta^{\alpha-1}\mathrm{d}\zeta \\ &-\frac{\alpha}{\pi}\int\limits_{\Omega^+} w_{\bar\zeta}(\zeta)\left[\frac{1}{\zeta^\alpha - z^\alpha} + \frac{z^\alpha}{z^\alpha\zeta^\alpha - 1}\right]\zeta^{\alpha-1}\mathrm{d}\xi\mathrm{d}\eta = \begin{cases} w(z), & z \in \Omega^+, \\ 0, & z \notin \overline{\Omega^+}, \end{cases}\end{aligned} \tag{2.70}$$

*and for* $z \in \Omega^+$

$$\begin{aligned} w(z) &= \frac{\alpha}{2\pi i}\int\limits_{\Gamma_0} \mathrm{Re}w(\zeta)\left(\frac{\zeta^\alpha + z^\alpha}{\zeta^\alpha - z^\alpha} - \frac{\overline{\zeta^\alpha} + z^\alpha}{\overline{\zeta^\alpha} - z^\alpha}\right)\frac{\mathrm{d}\zeta}{\zeta} + \frac{\alpha}{\pi}\int\limits_{\Gamma_0}\frac{\mathrm{Im}w(\zeta)}{\zeta}\mathrm{d}\zeta \\ &+\frac{\alpha}{\pi i}\int\limits_{[\varpi,0]\cup[0,1]} \mathrm{Re}w(\zeta)\left(\frac{1}{\zeta^\alpha - z^\alpha} - \frac{z^\alpha}{1 - z^\alpha\zeta^\alpha}\right)\zeta^{\alpha-1}\mathrm{d}\zeta \\ &-\frac{\alpha}{\pi}\int\limits_{\Omega^+}\Bigg[w_{\bar\zeta}(\zeta)\Big(\frac{1}{\zeta^\alpha - z^\alpha} - \frac{z^\alpha}{1 - z^\alpha\zeta^\alpha}\Big)\zeta^{\alpha-1} \\ &\qquad -\overline{w_{\bar\zeta}(\zeta)}\Big(\frac{1}{\overline{\zeta^\alpha} - z^\alpha} - \frac{z^\alpha}{1 - z^\alpha\overline{\zeta^\alpha}}\Big)\overline{\zeta^{\alpha-1}}\Bigg]\mathrm{d}\xi\mathrm{d}\eta. \end{aligned} \tag{2.71}$$

**Proof:** Let $\partial_{\bar z}w(z) = f(z)$ and denote a new function

$$W_0(z) = w(z^{1/\alpha}), \quad z \in \mathbb{D}^+,$$

then for $z \in \mathbb{D}^+$,

$$\begin{cases} \partial_{\bar z}W_0(z) = \dfrac{1}{\alpha}\,\overline{z^{\frac{1}{\alpha}-1}}\,f(z^{1/\alpha}), \\ \partial_z W_0(z) = \dfrac{1}{\alpha}\,z^{\frac{1}{\alpha}-1}\,\partial_z w(z^{1/\alpha}). \end{cases}$$

We observe that $W_0(z) \in C^1(\mathbb{D}^+;\mathbb{C}) \cap C(\overline{\mathbb{D}^+};\mathbb{C})$, thus from Lemma 2.4.1, we have for $z \in \Omega^+$,

$$\begin{aligned} W_0(z^\alpha) &= \frac{1}{2\pi i}\int\limits_{L_0} \mathrm{Re}W_0(\zeta)\left(\frac{\zeta + z^\alpha}{\zeta - z^\alpha} - \frac{\overline{\zeta} + z^\alpha}{\overline{\zeta} - z^\alpha}\right)\frac{\mathrm{d}\zeta}{\zeta} + \frac{1}{\pi}\int\limits_{L_0} \frac{\mathrm{Im}W_0(\zeta)}{\zeta}\mathrm{d}\zeta \\ &+\frac{1}{\pi i}\int\limits_{-1}^{1} \mathrm{Re}W_0(\zeta)\left(\frac{1}{\zeta - z^\alpha} - \frac{z^\alpha}{1 - z^\alpha\zeta}\right)\mathrm{d}\zeta \\ &-\frac{1}{\pi}\int\limits_{\mathbb{D}^+}\left[W_{0\bar{\zeta}}(\zeta)\Big(\frac{1}{\zeta - z^\alpha} - \frac{z^\alpha}{1 - z^\alpha\zeta}\Big) - \overline{W_{0\bar{\zeta}}(\zeta)}\Big(\frac{1}{\overline{\zeta} - z^\alpha} - \frac{z^\alpha}{1 - z^\alpha\overline{\zeta}}\Big)\right]\mathrm{d}\xi\mathrm{d}\eta \end{aligned}$$

where $W_{0\bar{\zeta}}(\zeta) = \dfrac{1}{\alpha}\,\overline{\zeta^{\frac{1}{\alpha}-1}}\, f(\zeta^{1/\alpha})$ and $L_0$ is defined in Lemma 2.4.1. Applying the transformation (2.68): $\zeta = \tau^\alpha$ with $\tau = \tau_1 + i\tau_2,\ \tau_1,\ \tau_2 \in \mathbb{R}$, thus, $\mathrm{d}\xi\mathrm{d}\eta = J\ \mathrm{d}\tau_1\mathrm{d}\tau_2$ with the Jacobi determinant

$$J = \alpha^2|\tau|^{2(\alpha-1)}.$$

Therefore,

$$\begin{aligned} W_0(z^\alpha) &= \frac{\alpha}{2\pi i}\int\limits_{\Gamma_0} \mathrm{Re}W_0(\tau^\alpha)\left(\frac{\tau^\alpha + z^\alpha}{\tau^\alpha - z^\alpha} - \frac{\overline{\tau^\alpha} + z^\alpha}{\overline{\tau^\alpha} - z^\alpha}\right)\frac{\mathrm{d}\tau}{\tau} + \frac{\alpha}{\pi}\int\limits_{\Gamma_0} \frac{\mathrm{Im}W_0(\tau^\alpha)}{\tau}\mathrm{d}\tau \\ &+\frac{\alpha}{\pi i}\int\limits_{0}^{1} \mathrm{Re}W_0(\tau^\alpha)\left(\frac{1}{\tau^\alpha - z^\alpha} - \frac{z^\alpha}{1 - z^\alpha\tau^\alpha}\right)\tau^{\alpha-1}\mathrm{d}\tau \\ &+\frac{\alpha}{\pi i}\int\limits_{[\varpi,0]} \mathrm{Re}W_0(\tau^\alpha)\left(\frac{1}{\tau^\alpha - z^\alpha} - \frac{z^\alpha}{1 - z^\alpha\tau^\alpha}\right)\tau^{\alpha-1}\mathrm{d}\tau \\ &-\frac{1}{\alpha\pi}\int\limits_{\Omega^+}\Bigg\{\Big[\overline{\tau^{1-\alpha}}f(\tau)\Big(\frac{1}{\tau^\alpha - z^\alpha} - \frac{z^\alpha}{1 - z^\alpha\tau^\alpha}\Big) \\ &\qquad -\tau^{1-\alpha}\overline{f(\tau)}\Big(\frac{1}{\overline{\tau^\alpha} - z^\alpha} - \frac{z^\alpha}{1 - z^\alpha\overline{\tau^\alpha}}\Big)\Big] \times \alpha^2|\tau|^{2(\alpha-1)}\Bigg\}\mathrm{d}\tau_1\mathrm{d}\tau_2. \end{aligned}$$

By simple computation, this is the desired conclusion (2.71). Similarly, expression (2.70) is also true.

**Remark 2.4.2.** We can verify that when $\alpha = n$ ($n \in \mathbb{N}$), the Schwarz-Poisson representation formula in Theorem 2.4.1 is just the result in Theorem 2.2.1.

In particular, the investigation for $\alpha = 1/2$ should be especially noted. For the case $\alpha = 1/2$, the fan-shaped domain with angle $2\pi$ is just

$$\mathbb{D}_0 = \{z,\ |z| < 1, z \notin [0,1)\} \ \text{ with } \ \partial\mathbb{D}_0 = [0,1] \cup \Gamma \cup [1,0],$$

where

$$\Gamma = \{\tau,\ |\tau| = 1,\ \tau \neq 1\},$$

is oriented counter clockwise. Let the unit disc

$$\mathbb{D} = \{z : |z| < 1\} \ \text{ with } \ \partial\mathbb{D} = \{\tau, |\tau| = 1\}.$$

Then we observe that

$$\mathbb{D}_0 = \mathbb{D}\backslash[0,1), \quad \Gamma = \partial\mathbb{D}\backslash\{1\}. \tag{2.72}$$

From Theorem 2.4.1, the result for $\alpha = 1/2$ holds.

**Corollary 2.4.1.** *Any $w \in C^1(\mathbb{D}_0;\mathbb{C}) \cap C(\overline{\mathbb{D}_0};\mathbb{C})$ can be represented as*

$$\begin{aligned}&\frac{1}{4\pi i}\int\limits_{\partial\mathbb{D}_0} w(\zeta)\left[\frac{1}{\zeta^{1/2}-z^{1/2}} + \frac{z^{1/2}}{z^{1/2}\zeta^{1/2}-1}\right]\zeta^{-1/2}\mathrm{d}\zeta \\ &-\frac{1}{2\pi}\int\limits_{\mathbb{D}_0} w_{\bar\zeta}(\zeta)\left[\frac{1}{\zeta^{1/2}-z^{1/2}} + \frac{z^{1/2}}{z^{1/2}\zeta^{1/2}-1}\right]\zeta^{-1/2}\mathrm{d}\xi\mathrm{d}\eta = \begin{cases} w(z), & z \in \mathbb{D}_0, \\ 0, & z \notin \overline{\mathbb{D}_0}, \end{cases}\end{aligned} \tag{2.73}$$

*and for $z \in \mathbb{D}_0$*

$$\begin{aligned} w(z) \ &= \frac{1}{4\pi i}\int\limits_{\Gamma} \mathrm{Re}w(\zeta)\left(\frac{\zeta^{1/2}+z^{1/2}}{\zeta^{1/2}-z^{1/2}} - \frac{\overline{\zeta^{1/2}}+z^{1/2}}{\overline{\zeta^{1/2}}-z^{1/2}}\right)\frac{\mathrm{d}\zeta}{\zeta} + \frac{1}{2\pi}\int\limits_{\Gamma}\frac{\mathrm{Im}w(\zeta)}{\zeta}\mathrm{d}\zeta \\ &+\frac{1}{2\pi i}\int\limits_{[1,0]\cup[0,1]} \mathrm{Re}w(\zeta)\left(\frac{1}{\zeta^{1/2}-z^{1/2}} - \frac{z^{1/2}}{1-z^{1/2}\zeta^{1/2}}\right)\zeta^{-1/2}\mathrm{d}\zeta \\ &-\frac{1}{2\pi}\int\limits_{\mathbb{D}_0}\Bigg[w_{\bar\zeta}(\zeta)\Big(\frac{1}{\zeta^{1/2}-z^{1/2}} - \frac{z^{1/2}}{1-z^{1/2}\zeta^{1/2}}\Big)\zeta^{-\frac{1}{2}} \\ &\qquad\qquad -\overline{w_{\bar\zeta}(\zeta)}\Big(\frac{1}{\overline{\zeta^{1/2}}-z^{1/2}} - \frac{z^{1/2}}{1-z^{1/2}\overline{\zeta^{1/2}}}\Big)\overline{\zeta^{-1/2}}\Bigg]\mathrm{d}\xi\mathrm{d}\eta, \end{aligned} \tag{2.74}$$

*where for $\zeta_+ \in [0,1]$, $\zeta_- \in [1,0]$ and $\zeta_+ = \zeta_-$, we have $\mathrm{Re}w(\zeta_-) \not\equiv \mathrm{Re}w(\zeta_+)$, as well as $w(\zeta)$.*

Actually, the classical Schwarz-Poisson formula for the unit disc ( see [7] ) can be derived from Corollary 2.4.1.

**Proposition 2.4.1.** *Any* $w \in C^1(\mathbb{D};\mathbb{C}) \cap C(\overline{\mathbb{D}};\mathbb{C})$ *is represented as*

$$\begin{aligned} w(z) \ &= \frac{1}{2\pi i}\int\limits_{|\zeta|=1} \mathrm{Re}w(\zeta)\frac{\zeta+z}{\zeta-z}\frac{d\zeta}{\zeta} + \frac{1}{2\pi}\int\limits_{|\zeta|=1} \mathrm{Im}w(\zeta)\frac{d\zeta}{\zeta} \\ &-\frac{1}{\pi}\int\limits_{|\zeta|<1}\left(\frac{w_{\bar\zeta}(\zeta)}{\zeta-z}+\frac{z\overline{w_{\bar\zeta}(\zeta)}}{1-z\bar\zeta}\right)d\xi d\eta, \quad |z|<1. \end{aligned} \tag{2.75}$$

**Proof:** Obviously, when $w \in C^1(\mathbb{D};\mathbb{C}) \cap C(\overline{\mathbb{D}};\mathbb{C})$, we have $w \in C^1(\mathbb{D}_0;\mathbb{C}) \cap C(\overline{\mathbb{D}_0};\mathbb{C})$. Moreover, $w(\zeta_+) = w(\zeta_-)$ for $\zeta_+ \in [0,1]$, $\zeta_- \in [1,0]$ and $\zeta_+ = \zeta_-$. Thus the third integral on the right-hand side in (2.74) equals 0. Then from (2.72) and (2.74), $w(z)$ is expressed as

$$\begin{aligned} w(z) \ &= \frac{1}{4\pi i}\int\limits_{\partial\mathbb{D}} \mathrm{Re}w(\zeta)\left(\frac{\zeta^{1/2}+z^{1/2}}{\zeta^{1/2}-z^{1/2}} - \frac{\overline{\zeta^{1/2}}+z^{1/2}}{\overline{\zeta^{1/2}}-z^{1/2}}\right)\frac{d\zeta}{\zeta} + \frac{1}{2\pi}\int\limits_{\partial\mathbb{D}}\frac{\mathrm{Im}w(\zeta)}{\zeta}d\zeta \\ &-\frac{1}{2\pi}\int\limits_{\mathbb{D}}\Bigg[w_{\bar\zeta}(\zeta)\Big(\frac{1}{\zeta^{1/2}-z^{1/2}} - \frac{z^{1/2}}{1-z^{1/2}\zeta^{1/2}}\Big)\zeta^{-\frac{1}{2}} \\ &\qquad -\overline{w_{\bar\zeta}(\zeta)}\Big(\frac{1}{\overline{\zeta^{1/2}}-z^{1/2}} - \frac{z^{1/2}}{1-z^{1/2}\overline{\zeta^{1/2}}}\Big)\overline{\zeta^{-1/2}}\Bigg]d\xi d\eta. \end{aligned} \tag{2.76}$$

Since

$$\begin{aligned} \frac{1}{\zeta-z} &= \frac{1}{2}\zeta^{-1/2}\left(\frac{1}{\zeta^{1/2}-z^{1/2}}+\frac{1}{\zeta^{1/2}+z^{1/2}}\right), \\ \frac{z}{1-z\bar\zeta} &= \frac{1}{2}z^{1/2}\overline{\zeta^{-1/2}}\left(\frac{1}{1-z^{1/2}\overline{\zeta^{1/2}}}-\frac{1}{1+z^{1/2}\overline{\zeta^{1/2}}}\right), \end{aligned}$$

therefore, the area integral in (2.76) can be converted into

$$\begin{aligned} &-\frac{1}{\pi}\int\limits_{\mathbb{D}}\left(\frac{w_{\bar\zeta}(\zeta)}{\zeta-z}+\frac{z\overline{w_{\bar\zeta}(\zeta)}}{1-z\bar\zeta}\right)d\xi d\eta + \frac{1}{2\pi}\int\limits_{\mathbb{D}}\Bigg\{w_{\bar\zeta}(\zeta)\Bigg[\frac{\zeta^{-1/2}}{\zeta^{1/2}+z^{1/2}} \\ &\quad +\frac{z^{1/2}\zeta^{-1/2}}{1-z^{1/2}\zeta^{1/2}}\Bigg] + \overline{w_{\bar\zeta}(\zeta)}\Bigg[\frac{\overline{\zeta^{-1/2}}}{\overline{\zeta^{1/2}}-z^{1/2}} - \frac{z^{1/2}\overline{\zeta^{-1/2}}}{1+z^{1/2}\overline{\zeta^{1/2}}}\Bigg]\Bigg\}d\xi d\eta. \end{aligned} \tag{2.77}$$

Let $\Delta_0$ be the second area integral in (2.77), then from the Gauss theorem,

$$\begin{aligned}\Delta_0 &= \lim_{\varepsilon\to 0^+}\frac{1}{4\pi i}\left(\int\limits_{|\zeta|=1}-\int\limits_{|\zeta|=\varepsilon}\right)\left\{w(\zeta)\left[\frac{\zeta^{1/2}}{\zeta^{1/2}+z^{1/2}}+\frac{z^{1/2}\zeta^{1/2}}{1-z^{1/2}\zeta^{1/2}}\right]\right.\\ &\qquad\left.+\overline{w(\zeta)}\left[\frac{\overline{\zeta^{1/2}}}{\overline{\zeta^{1/2}}-z^{1/2}}-\frac{z^{1/2}\overline{\zeta^{1/2}}}{1+z^{1/2}\overline{\zeta^{1/2}}}\right]\right\}\frac{d\zeta}{\zeta} \qquad (2.78)\\ &= \frac{1}{4\pi i}\int\limits_{|\zeta|=1}\mathrm{Re}w(\zeta)\left[\frac{\zeta^{1/2}-z^{1/2}}{\zeta^{1/2}+z^{1/2}}+\frac{1+z^{1/2}\zeta^{1/2}}{1-z^{1/2}\zeta^{1/2}}\right]\frac{d\zeta}{\zeta}.\end{aligned}$$

Thus, by (2.76)-(2.78),

$$\begin{aligned}w(z) &= \frac{1}{2\pi}\int\limits_{|\zeta|=1}\frac{\mathrm{Im}w(\zeta)}{\zeta}d\zeta-\frac{1}{\pi}\int\limits_{|\zeta|<1}\left(\frac{w_{\bar\zeta}(\zeta)}{\zeta-z}+\frac{z\overline{w_{\bar\zeta}(\zeta)}}{1-z\bar\zeta}\right)d\xi d\eta\\ &\quad+\frac{1}{4\pi i}\int\limits_{|\zeta|=1}\mathrm{Re}w(\zeta)\left(\frac{\zeta^{1/2}+z^{1/2}}{\zeta^{1/2}-z^{1/2}}-\frac{\overline{\zeta^{1/2}}+z^{1/2}}{\overline{\zeta^{1/2}}-z^{1/2}}\right.\\ &\qquad\left.+\frac{\zeta^{1/2}-z^{1/2}}{\zeta^{1/2}+z^{1/2}}+\frac{1+z^{1/2}\zeta^{1/2}}{1-z^{1/2}\zeta^{1/2}}\right)\frac{d\zeta}{\zeta}.\end{aligned}$$

By simple computation, (2.75) is valid. Then, the proof is completed.

### 2.4.2 Schwarz Problem

In this part, we consider the Schwarz problem in the fan-shaped domain $\Omega^+$ with angle $\pi/\alpha$ ($\alpha \geq 1/2$).

**Schwarz boundary value problem** Find a function $w$ satisfying the following conditions

$$\begin{cases} w_{\bar z}=f \text{ in } \Omega^+, \quad \mathrm{Re}w=\gamma \text{ on } \partial\Omega^+,\\ \dfrac{\alpha}{\pi}\displaystyle\int_0^{\frac{\pi}{\alpha}}\mathrm{Im}w(e^{i\varphi})d\varphi=c, \quad c\in\mathbb{R},\end{cases} \qquad (2.79)$$

with $f\in L_p(\Omega^+;\mathbb{C}),\ p>2,\ \ \gamma\in C(\partial\Omega^+;\mathbb{R})$.

Firstly, introducing a new kernel

$$H(z,\zeta)=\left[\frac{1}{\zeta^\alpha-z^\alpha}-\frac{1}{\zeta^\alpha-\overline{z^\alpha}}-\frac{z^\alpha}{1-z^\alpha\zeta^\alpha}+\frac{\overline{z^\alpha}}{1-\overline{z^\alpha}\zeta^\alpha}\right]\zeta^{\alpha-1}. \qquad (2.80)$$

Then, the following lemmas are valid.

**Lemma 2.4.2.** *If* $\gamma(\zeta) \in C(\partial\Omega^+; \mathbb{C})$, *then*

$$\lim_{z\in\Omega^+,\ z\to 1} \frac{\alpha}{2\pi i} \int_{\Gamma_0} [\gamma(\zeta) - \gamma(1)] H(z,\zeta) \mathrm{d}\zeta = 0.$$

**Proof:** By simple computation,

$$\begin{aligned}
&\lim_{z\in\Omega^+,\ z\to 1} \frac{\alpha}{2\pi i} \int_{\Gamma_0} [\gamma(\zeta) - \gamma(1)] H(z,\zeta) \mathrm{d}\zeta \\
&= \lim_{z\in\Omega^+,\ z\to 1} \left\{ \frac{1}{2\pi i} \int_{L_0} [\gamma(\zeta^{1/\alpha}) - \gamma(1)] \left[ \frac{\zeta}{\zeta - z^\alpha} + \frac{\overline{\zeta}}{\overline{\zeta} - \overline{z^\alpha}} - 1 \right] \frac{\mathrm{d}\zeta}{\zeta} \right. \\
&\quad \left. + \frac{1}{2\pi i} \int_{\widetilde{L}_0} [-\gamma(\overline{\zeta}^{1/\alpha}) + \gamma(1)] \left[ \frac{\zeta}{\zeta - z^\alpha} + \frac{\overline{\zeta}}{\overline{\zeta} - \overline{z^\alpha}} - 1 \right] \frac{\mathrm{d}\zeta}{\zeta} \right\} \\
&= \lim_{z\in\Omega^+,\ z\to 1} \frac{1}{2\pi i} \int_{|\zeta|=1} \Gamma_1(\zeta) \left[ \frac{\zeta}{\zeta - z^\alpha} + \frac{\overline{\zeta}}{\overline{\zeta} - \overline{z^\alpha}} - 1 \right] \frac{\mathrm{d}\zeta}{\zeta},
\end{aligned}$$

where $\widetilde{L}_0 = \{\tau : \ |\tau| = 1, \ \mathrm{Im}\tau < 0\}$ is oriented counter-clockwise and

$$\Gamma_1(\zeta) = \begin{cases} \gamma(\zeta^{1/\alpha}) - \gamma(1), & \zeta \in L_0, \\ -\gamma(\overline{\zeta}^{1/\alpha}) + \gamma(1), & \zeta \in \widetilde{L}_0. \end{cases}$$

Therefore, from the continuity of $\Gamma_1(\zeta)$ at $\zeta = 1$ and the property of the Poisson operator on the unit circle,

$$\lim_{z\in\Omega^+,\ z\to 1} \frac{\alpha}{2\pi i} \int_{\Gamma_0} [\gamma(\zeta) - \gamma(1)] H(z,\zeta) \mathrm{d}\zeta = \Gamma_1(z^\alpha) \big|_{z=1} = 0.$$

**Lemma 2.4.3.** *With* $\gamma(\zeta) \in C(\partial\Omega^+; \mathbb{C})$,

$$\lim_{z\in\Omega^+,\ z\to 1} \frac{\alpha}{2\pi i} \int_{[0,1]\cup[\varpi,0]} [\gamma(\zeta) - \gamma(1)] H(z,\zeta) \mathrm{d}\zeta = 0.$$

**Proof:** We observe that

$$\begin{aligned}
&\lim_{z\in\Omega^+,\ z\to 1}\frac{\alpha}{2\pi i}\int\limits_0^1[\gamma(\zeta)-\gamma(1)]H(z,\zeta)\mathrm{d}\zeta\\
&=\lim_{z\in\Omega^+,\ z\to 1}\left\{\frac{1}{2\pi i}\int_0^1[\gamma(\zeta^{1/\alpha})-\gamma(1)]\left[\frac{1}{\zeta-z^\alpha}-\frac{1}{\zeta-\overline{z^\alpha}}\right]\mathrm{d}\zeta\right.\\
&\qquad\left.+\frac{1}{2\pi i}\int_1^{+\infty}\left[-\gamma\left((\frac{1}{\zeta})^{1/\alpha}\right)+\gamma(1)\right]\left[\frac{1}{\zeta-z^\alpha}-\frac{1}{\zeta-\overline{z^\alpha}}\right]\mathrm{d}\zeta\right\}\\
&=\lim_{z\in\Omega^+,\ z\to 1}\frac{1}{2\pi i}\int_0^{+\infty}\Gamma_2(\zeta)\left[\frac{1}{\zeta-z^\alpha}-\frac{1}{\zeta-\overline{z^\alpha}}\right]\mathrm{d}\zeta,
\end{aligned}$$

where

$$\Gamma_2(\zeta)=\begin{cases}\gamma(\zeta^{1/\alpha})-\gamma(1), & \zeta\in(0,1],\\ -\gamma\left((\frac{1}{\zeta})^{1/\alpha}\right)+\gamma(1), & \zeta\in(1,+\infty).\end{cases}$$

Hence, similarly from the continuity of $\Gamma_2(\zeta)$ at $\zeta=1$ and the property of the Poisson operator on the real axis,

$$\lim_{z\in\Omega^+,\ z\to 1}\frac{\alpha}{2\pi i}\int_0^1[\gamma(\zeta)-\gamma(1)]H(z,\zeta)\mathrm{d}\zeta=\Gamma_2(1)=0.$$

Thus, the desired conclusion is obtained also from $H(z,\zeta)=0$ for $(z,\zeta)\in\{1\}\times[\varpi,0]$.

**Lemma 2.4.4.** *With* $\gamma(\zeta)\in C(\partial\Omega^+;\mathbb{C})$,

$$\lim_{z\in\Omega^+,\ z\to\varpi}\frac{\alpha}{2\pi i}\int\limits_{\Gamma_0}[\gamma(\zeta)-\gamma(\varpi)]H(z,\zeta)\mathrm{d}\zeta=0.$$

**Proof:** Similarly to Lemma 2.4.2,

$$\begin{aligned}
&\lim_{z\in\Omega^+,\ z\to\varpi}\frac{\alpha}{2\pi i}\int\limits_{\Gamma_0}[\gamma(\zeta)-\gamma(\varpi)]H(z,\zeta)\mathrm{d}\zeta\\
&=\lim_{z\in\Omega^+,\ z\to\varpi}\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\Gamma_3(\zeta)\left[\frac{\zeta}{\zeta-z^\alpha}+\frac{\overline{\zeta}}{\overline{\zeta}-\overline{z^\alpha}}-1\right]\frac{\mathrm{d}\zeta}{\zeta},
\end{aligned}$$

where

$$\Gamma_3(\zeta)=\begin{cases}\gamma(\zeta^{1/\alpha})-\gamma(\varpi), & \zeta\in L_0,\\ -\gamma(\overline{\zeta}^{1/\alpha})+\gamma(\varpi), & \zeta\in\widetilde{L}_0.\end{cases}$$

Since when $z \to \varpi$, $z^\alpha$ tends to $-1$ and $\Gamma_3$ is continuous at $\zeta = z^\alpha$ ($z = \varpi$), thus,

$$\lim_{z\in\Omega^+,\ z\to\varpi} \frac{\alpha}{2\pi i} \int\limits_{\Gamma_0} [\gamma(\zeta) - \gamma(\varpi)] H(z,\zeta) \mathrm{d}\zeta = \Gamma_3(z^\alpha)\big|_{z=\varpi} = 0.$$

**Lemma 2.4.5.** *With* $\gamma(\zeta) \in C(\partial\Omega^+;\mathbb{C})$,

$$\lim_{z\in\Omega^+,\ z\to\varpi} \frac{\alpha}{2\pi i} \int\limits_{[0,1]\cup[\varpi,0]} [\gamma(\zeta) - \gamma(\varpi)] H(z,\zeta) \mathrm{d}\zeta = 0.$$

**Proof:** Since $H(z,\zeta) = 0$ for $(z,\zeta) \in \{\varpi\} \times [0,1]$, then

$$\begin{aligned} &\lim_{z\in\Omega^+,\ z\to\varpi} \frac{\alpha}{2\pi i} \int\limits_{[0,1]\cup[\varpi,0]} [\gamma(\zeta) - \gamma(\varpi)] H(z,\zeta) \mathrm{d}\zeta \\ &= \lim_{z\in\Omega^+,\ z\to\varpi} \frac{1}{2\pi i} \int_{-\infty}^{0} \Gamma_4(\zeta) \left[\frac{1}{\zeta - z^\alpha} - \frac{1}{\zeta - \overline{\overline{z^\alpha}}}\right] \mathrm{d}\zeta, \end{aligned}$$

where

$$\Gamma_4(\zeta) = \begin{cases} \gamma(\zeta^{1/\alpha}) - \gamma(\varpi), & \zeta \in [-1,0), \\ -\gamma\left(\left(\frac{1}{\zeta}\right)^{1/\alpha}\right) + \gamma(\varpi), & \zeta \in (-\infty,-1). \end{cases}$$

Therefore, similarly from the continuity of $\Gamma_4(\zeta)$ at $\zeta = z^\alpha$ ($z = \varpi$), we obtain,

$$\lim_{z\in\Omega^+,\ z\to\varpi} \frac{\alpha}{2\pi i} \int\limits_{[0,1]\cup[\varpi,0]} [\gamma(\zeta) - \gamma(\varpi)] H(z,\zeta) \mathrm{d}\zeta = \Gamma_4(z^\alpha)\big|_{z=\varpi} = 0.$$

**Lemma 2.4.6.** *With* $\gamma(\zeta) \in C(\partial\Omega^+;\mathbb{C})$,

$$\lim_{z\in\Omega^+,\ z\to 0} \frac{\alpha}{2\pi i} \int\limits_{\partial\Omega^+} [\gamma(\zeta) - \gamma(0)] H(z,\zeta) \mathrm{d}\zeta = 0.$$

**Proof:** By $H(z,\zeta) = 0$ for $(z,\zeta) \in \{0\} \times \Gamma_0$, we see that

$$\begin{aligned} &\lim_{z\in\Omega^+,\ z\to 0} \frac{\alpha}{2\pi i} \int\limits_{\partial\Omega^+} [\gamma(\zeta) - \gamma(0)] H(z,\zeta) \mathrm{d}\zeta \\ &= \lim_{z\in\Omega^+,\ z\to 0} \left\{ \frac{1}{2\pi i} \int_0^1 [\gamma(\zeta^{1/\alpha}) - \gamma(0)] \left[\frac{1}{\zeta - z^\alpha} - \frac{1}{\zeta - \overline{\overline{z^\alpha}}}\right] \mathrm{d}\zeta \right. \\ &\quad \left. + \frac{1}{2\pi i} \int_{-1}^0 [\gamma(\zeta^{1/\alpha}) - \gamma(0)] \left[\frac{1}{\zeta - z^\alpha} - \frac{1}{\zeta - \overline{\overline{z^\alpha}}}\right] \mathrm{d}\zeta \right\} \\ &= \lim_{z\in\Omega^+,\ z\to 0} \frac{1}{2\pi i} \int_{-1}^1 [\gamma(\zeta^{1/\alpha}) - \gamma(0)] \left[\frac{1}{\zeta - z^\alpha} - \frac{1}{\zeta - \overline{\overline{z^\alpha}}}\right] \mathrm{d}\zeta = [\gamma(z) - \gamma(0)]\big|_{z=0} = 0. \end{aligned}$$

**Remark 2.4.3.** Lemmas 2.4.2-2.4.6 suggest that the following boundary behavior is valid at the three corner points, that is,

$$\lim_{z\in\Omega^+,\ z\to t}\frac{\alpha}{2\pi i}\int\limits_{\partial\Omega^+}[\gamma(\zeta)-\gamma(t)]H(z,\zeta)\mathrm{d}\zeta=0,\quad t\in\{0,1,\varpi\}.$$

According to the representation formula in Theorem 2.4.1, the next result is true.

**Theorem 2.4.2.** *The Schwarz problem* (2.79) *for the domain* $\Omega^+$ *is uniquely solvable by*

$$\begin{aligned} w(z) &= \frac{\alpha}{2\pi i}\int\limits_{\Gamma_0}\gamma(\zeta)\left(\frac{\zeta^\alpha+z^\alpha}{\zeta^\alpha-z^\alpha}-\frac{\overline{\zeta^\alpha}+z^\alpha}{\overline{\zeta^\alpha}-z^\alpha}\right)\frac{\mathrm{d}\zeta}{\zeta}+ic \\ &+\frac{\alpha}{\pi i}\int\limits_{[\varpi,0]\cup[0,1]}\gamma(\zeta)\left(\frac{1}{\zeta^\alpha-z^\alpha}-\frac{z^\alpha}{1-z^\alpha\zeta^\alpha}\right)\zeta^{\alpha-1}\mathrm{d}\zeta \\ &-\frac{\alpha}{\pi}\int\limits_{\Omega^+}\Bigg[f(\zeta)\Big(\frac{1}{\zeta^\alpha-z^\alpha}-\frac{z^\alpha}{1-z^\alpha\zeta^\alpha}\Big)\zeta^{\alpha-1} \\ &\quad -\overline{f(\zeta)}\Big(\frac{1}{\overline{\zeta^\alpha}-z^\alpha}-\frac{z^\alpha}{1-z^\alpha\overline{\zeta^\alpha}}\Big)\overline{\zeta^{\alpha-1}}\Bigg]\mathrm{d}\xi\mathrm{d}\eta,\quad z\in\Omega. \end{aligned} \tag{2.81}$$

**Proof:** From Theorem 2.4.1, we only need to verify that (2.81) provides a solution. Since for $z,\ \zeta\in\Omega^+$,

$$\lim_{\zeta\longrightarrow z}(\zeta-z)\frac{\alpha\zeta^{\alpha-1}}{\zeta^\alpha-z^\alpha}=1,$$

then we know that $z$ is a simple pole of $\dfrac{\alpha\zeta^{\alpha-1}}{\zeta^\alpha-z^\alpha}$ in $\Omega^+$ and

$$\frac{\alpha\zeta^{\alpha-1}}{\zeta^\alpha-z^\alpha}=\frac{1}{\zeta-z}+g(z,\zeta),$$

where for arbitrary $z\in\Omega^+$, $g(z,\zeta)$ is analytic with respect to $z$. Thus, we obviously see that

$$\partial_{\overline{z}}w(z)=\partial_{\overline{z}}\left\{-\frac{1}{\pi}\int\limits_{\Omega^+}\frac{f(\zeta)}{\zeta-z}\mathrm{d}\xi\mathrm{d}\eta\right\}=f(z).$$

When $\zeta \in \Gamma_0$, i.e, $|\zeta| = 1$,

$$\frac{\alpha}{2\pi i}\int\limits_{\Gamma_0}\left(\frac{\zeta^\alpha + z^\alpha}{\zeta^\alpha - z^\alpha} - \frac{\overline{\zeta^\alpha} + z^\alpha}{\overline{\zeta^\alpha} - z^\alpha}\right)\frac{\mathrm{dz}}{z} = \frac{1}{2\pi i}\int\limits_{L_0}\left(\frac{\zeta^\alpha + z}{\zeta^\alpha - z} + \frac{\zeta^\alpha + \overline{z}}{\zeta^\alpha - \overline{z}}\right)\frac{\mathrm{dz}}{z}$$
$$= \frac{1}{2\pi i}\int\limits_{|z|=1}\frac{\zeta^\alpha + z}{\zeta^\alpha - z}\frac{\mathrm{dz}}{z} = 0,$$

and for $\zeta \in [0,1) \cup (\varpi, 0] \cup \Omega^+$,

$$\frac{\alpha}{2\pi i}\int\limits_{\Gamma_0}\Big(\frac{1}{\zeta^\alpha - z^\alpha} - \frac{z^\alpha}{1 - z^\alpha\zeta^\alpha}\Big)\frac{\mathrm{dz}}{z} = \frac{1}{2\pi i}\int\limits_{L_0}\Big(\frac{1}{\zeta^\alpha - z} + \frac{1}{\zeta^\alpha - \overline{z}}\Big)\frac{\mathrm{dz}}{z}$$
$$= \frac{1}{2\pi i}\int\limits_{|z|=1}\frac{1}{\zeta^\alpha - z}\frac{\mathrm{dz}}{z} = 0.$$

Similarly, for $\zeta \in \Omega^+$,

$$\frac{\alpha}{2\pi i}\int\limits_{\Gamma_0}\left(\frac{1}{\overline{\zeta^\alpha} - z^\alpha} - \frac{z^\alpha}{1 - z^\alpha\overline{\zeta^\alpha}}\right)\frac{\mathrm{dz}}{z} = 0.$$

Thus,

$$\frac{\alpha}{\pi}\int_0^{\frac{\pi}{\alpha}} \mathrm{Im}w(e^{i\varphi})\mathrm{d}\varphi = \frac{\alpha}{\pi}\int_0^{\frac{\pi}{\alpha}} c\mathrm{d}\varphi = c.$$

Let $w_0$ be the area integral in (2.81), then

$$\mathrm{Re}w_0(z) = -\frac{\alpha}{2\pi}\int\limits_{\Omega^+}\left[f(\zeta)H(z,\zeta) + \overline{f(\zeta)}\ \overline{H(z,\zeta)}\right]\mathrm{d}\xi\mathrm{d}\eta.$$

Since $H(z,\zeta) = 0$ for $(z,\zeta) \in \partial\Omega^+ \times \Omega^+$, we obtain $\mathrm{Re}w_0(z) = 0$ for $z \in \partial\Omega^+$. We can write $\mathrm{Re}w(z)$ as

$$\mathrm{Re}w(z) = \frac{\alpha}{2\pi i}\int\limits_{\partial\Omega^+}\gamma(\zeta)H(z,\zeta)\mathrm{d}\zeta + \mathrm{Re}w_0(z). \tag{2.82}$$

Since $H(z,\zeta) = 0$ for $(z,\zeta) \in \big\{\Gamma_0\backslash\{1,\varpi\}\big\} \times \{[\varpi,0] \cup [0,1]\}$, then for $\zeta_0 \in \Gamma_0\backslash\{1,\ \varpi\}$,

$$\lim_{z\to\zeta_0}\frac{\alpha}{2\pi i}\int\limits_{\partial\Omega^+}\gamma(\zeta)H(z,\zeta)\mathrm{d}\zeta = \lim_{z\to\zeta_0}\frac{\alpha}{2\pi i}\int\limits_{\Gamma_0}\gamma(\zeta)\left[\frac{1-|z^\alpha|^2}{|\zeta^\alpha - z^\alpha|^2} - \frac{1-|z^\alpha|^2}{|\zeta^\alpha - \overline{z}^\alpha|^2}\right]\frac{\mathrm{d}\zeta}{\zeta}$$
$$= \lim_{z\to\zeta_0}\frac{\alpha}{2\pi i}\int\limits_{\Gamma_0}\gamma(\zeta)\frac{1-|z^\alpha|^2}{|\zeta^\alpha - z^\alpha|^2}\frac{\mathrm{d}\zeta}{\zeta} = \lim_{z\to\zeta_0}\frac{1}{2\pi i}\int\limits_{L_0}\gamma(\zeta^{\frac{1}{\alpha}})\frac{1-|z^\alpha|^2}{|\zeta - z^\alpha|^2}\frac{\mathrm{d}\zeta}{\zeta} = \gamma(\zeta_0).$$

Similarly, $H(z,\zeta)=0$ for $(z,\zeta)\in(0,1)\times\{[\varpi,0]\cup\Gamma_0\}$, then for $\zeta_0\in(0,1)$,

$$\lim_{z\to\zeta_0}\frac{\alpha}{2\pi i}\int\limits_{\partial\Omega^+}\gamma(\zeta)H(z,\zeta)\mathrm{d}\zeta=\lim_{z\to\zeta_0}\frac{\alpha}{2\pi i}\int\limits_{0}^{1}\gamma(\zeta)\left[\frac{z^\alpha-\overline{z^\alpha}}{|\zeta^\alpha-z^\alpha|^2}-\frac{z^\alpha-\overline{z^\alpha}}{|1-z^\alpha\zeta^\alpha|^2}\right]\zeta^{\alpha-1}\mathrm{d}\zeta$$

$$=\lim_{z\to\zeta_0}\frac{\alpha}{2\pi i}\int\limits_{0}^{1}\gamma(\zeta)\frac{z^\alpha-\overline{z^\alpha}}{|\zeta^\alpha-z^\alpha|^2}\zeta^{\alpha-1}\mathrm{d}\zeta=\lim_{z\to\zeta_0}\frac{1}{2\pi i}\int\limits_{0}^{1}\gamma(\zeta^{\frac{1}{\alpha}})\frac{z^\alpha-\overline{z^\alpha}}{|\zeta-z^\alpha|^2}\mathrm{d}\zeta=\gamma(\zeta_0).$$

Moreover, when $(z,\zeta)\in(\varpi,0)\times\{[0,1]\cup\Gamma_0\}$, $H(z,\zeta)=0$. Thus for $\zeta_0\in(\varpi,0)$,

$$\lim_{z\to\zeta_0}\frac{\alpha}{2\pi i}\int\limits_{\partial\Omega^+}\gamma(\zeta)H(z,\zeta)\mathrm{d}\zeta=\lim_{z\to\zeta_0}\frac{1}{2\pi i}\int\limits_{-1}^{0}\gamma(\zeta^{\frac{1}{\alpha}})\frac{z^\alpha-\overline{z^\alpha}}{|\zeta-z^\alpha|^2}\mathrm{d}\zeta=\gamma(\zeta_0).$$

In a word, we have $\lim\limits_{z\to\zeta_0}\mathrm{Re}w(z)=\gamma(\zeta_0)$ for $\zeta_0\in\partial\Omega^+$ except for the three corner points.

With respect to the three corner points, we can convert (2.82) into different representations. Firstly, $\mathrm{Re}w(z)$ can be expressed as

$$\mathrm{Re}w(z)\ =\frac{\alpha}{2\pi i}\int\limits_{\partial\Omega^+}\left[\gamma(\zeta)-\gamma(1)\right]H(z,\zeta)\mathrm{d}\zeta+\frac{\alpha\gamma(1)}{2\pi i}\int\limits_{\partial\Omega^+}H(z,\zeta)\mathrm{d}\zeta+\mathrm{Re}w_0(z). \tag{2.83}$$

Applying the representation (2.71) to $w(z)\equiv 1$ and then taking the real part on both sides, we obtain

$$\begin{aligned}\frac{\alpha}{2\pi i}\int\limits_{\partial\Omega^+}H(z,\zeta)\mathrm{d}\zeta=\frac{\alpha}{2\pi i}\int\limits_{\partial\Omega^+}&\left[\frac{1}{\zeta^\alpha-z^\alpha}-\frac{1}{\zeta^\alpha-\overline{z^\alpha}}\right.\\&\left.-\frac{z^\alpha}{1-z^\alpha\zeta^\alpha}+\frac{\overline{z^\alpha}}{1-\overline{z^\alpha}\zeta^\alpha}\right]\zeta^{\alpha-1}\mathrm{d}\zeta=1,\end{aligned} \tag{2.84}$$

then, also from Lemmas 2.4.2 and 2.4.3,

$$\lim_{z\in\Omega^+,\ z\to 1}\mathrm{Re}w(z)=\gamma(1).$$

Similarly as before,

$$\mathrm{Re}w(z)\ =\frac{\alpha}{2\pi i}\int\limits_{\partial\Omega^+}\left[\gamma(\zeta)-\gamma(\varpi)\right]H(z,\zeta)\mathrm{d}\zeta+\frac{\alpha\gamma(\varpi)}{2\pi i}\int\limits_{\partial\Omega^+}H(z,\zeta)\mathrm{d}\zeta+\mathrm{Re}w_0(z). \tag{2.85}$$

Hence, $\lim_{z\in\Omega^+,\ z\to\varpi} \mathrm{Re}w(z) = \gamma(\varpi)$ follows from (2.84), Lemma 2.4.4 and Lemma 2.4.5.

Also obviously, we can rewrite $\mathrm{Re}w(z)$ as,

$$\mathrm{Re}w(z) = \frac{\alpha}{2\pi i}\int\limits_{\partial\Omega^+} [\gamma(\zeta)-\gamma(0)]\, H(z,\zeta)\mathrm{d}\zeta + \frac{\alpha\gamma(0)}{2\pi i}\int\limits_{\partial\Omega^+} H(z,\zeta)\mathrm{d}\zeta + \mathrm{Re}w_0(z). \tag{2.86}$$

Thus, the desired result $\lim_{z\in\Omega^+,\ z\to 0} \mathrm{Re}w(z) = \gamma(0)$ is true from (2.84) and Lemma 2.4.6. Therefore, the proof is completed.

## 2.5 Dirichlet Problem with Angle $\pi/\alpha$ ($\alpha \geq 1/2$)

In this Section, a Dirichlet boundary value problem for the domain $\Omega^+$ is discussed.

**Theorem 2.5.1. Dirichlet boundary value problem**

$$\begin{cases} w_{\overline{z}} = f \ \ in\ \Omega^+, & f \in L_p(\Omega^+;\mathbb{C}),\ p>2, \\ w = \gamma \ \ on\ \partial\Omega^+, & \gamma \in C(\partial\Omega^+;\mathbb{C}), \end{cases}$$

*is solvable if and only if for* $z \in \Omega^+$,

$$\begin{aligned} &\frac{\alpha}{2\pi i}\int\limits_{\partial\Omega^+} \gamma(\zeta)\left(\frac{1}{\zeta^\alpha - \overline{z^\alpha}} + \frac{\overline{z^\alpha}}{\overline{z^\alpha}\zeta^\alpha - 1}\right)\zeta^{\alpha-1}\mathrm{d}\zeta \\ &= \frac{\alpha}{\pi}\int\limits_{\Omega^+} f(\zeta)\left(\frac{1}{\zeta^\alpha - \overline{z^\alpha}} + \frac{\overline{z^\alpha}}{\overline{z^\alpha}\zeta^\alpha - 1}\right)\zeta^{\alpha-1}\mathrm{d}\xi\mathrm{d}\eta. \end{aligned} \tag{2.87}$$

*Then the solution can uniquely be expressed as*

$$\begin{aligned} w(z) &= \frac{\alpha}{2\pi i}\int\limits_{\partial\Omega^+} \gamma(\zeta)\left(\frac{1}{\zeta^\alpha - z^\alpha} + \frac{z^\alpha}{z^\alpha\zeta^\alpha - 1}\right)\zeta^{\alpha-1}\mathrm{d}\zeta \\ &\quad - \frac{\alpha}{\pi}\int\limits_{\Omega^+} f(\zeta)\left(\frac{1}{\zeta^\alpha - z^\alpha} + \frac{z^\alpha}{z^\alpha\zeta^\alpha - 1}\right)\zeta^{\alpha-1}\mathrm{d}\xi\mathrm{d}\eta, \quad z\in\Omega^+. \end{aligned} \tag{2.88}$$

**Proof:** If there is a solution, it must be of the form (2.88) from Theorem 2.4.1. Let $w$ be the solution to the Dirichlet problem, then we know

$$\lim_{z\to\zeta} w(z) = \gamma(\zeta) \quad \text{for} \quad \zeta \in \partial\Omega^+. \tag{2.89}$$

We consider a new function

$$\begin{aligned} G(z) &= \frac{\alpha}{2\pi i} \int\limits_{\partial\Omega^+} \gamma(\zeta)\left(\frac{1}{\zeta^\alpha - \overline{z^\alpha}} + \frac{\overline{z^\alpha}}{\overline{z^\alpha}\zeta^\alpha - 1}\right)\zeta^{\alpha-1}\mathrm{d}\zeta \\ &\quad -\frac{\alpha}{\pi}\int\limits_{\Omega^+} f(\zeta)\left(\frac{1}{\zeta^\alpha - \overline{z^\alpha}} + \frac{\overline{z^\alpha}}{\overline{z^\alpha}\zeta^\alpha - 1}\right)\zeta^{\alpha-1}\mathrm{d}\xi\mathrm{d}\eta, \quad z \in \Omega^+. \end{aligned}$$

Thus, we have

$$w(z) - G(z) = \frac{\alpha}{2\pi i}\int\limits_{\partial\Omega^+} \gamma(\zeta)H(z,\zeta)\mathrm{d}\zeta - \frac{\alpha}{\pi}\int\limits_{\Omega^+} f(\zeta)H(z,\zeta)\mathrm{d}\xi\mathrm{d}\eta. \tag{2.90}$$

Since the area integral in (2.90) vanishes on $\partial\Omega^+$, then from the proof of Theorem 2.4.2, we easily obtain, for $\zeta \in \partial\Omega^+$,

$$\lim_{z\in\Omega^+,\ z\to\zeta} [w(z) - G(z)] = \gamma(\zeta), \tag{2.91}$$

which implies

$$\lim_{z\in\Omega^+,\ z\to\zeta} G(z) = 0.$$

We observe that $\overline{G(z)}$ is analytic for $z \in \Omega^+$, and then by the maximum principle for analytic functions, we know $G(z) \equiv 0$ for $z \in \Omega^+$, which is just the condition (2.87).

On the other hand, if the condition (2.87) is satisfied, then we can rewrite $w(z)$ as

$$w(z) = \frac{\alpha}{2\pi i}\int\limits_{\partial\Omega^+} \gamma(\zeta)H(z,\zeta)\mathrm{d}\zeta - \frac{\alpha}{\pi}\int\limits_{\Omega^+} f(\zeta)H(z,\zeta)\mathrm{d}\xi\mathrm{d}\eta.$$

Obviously, $\partial_{\overline{z}}w(z) = f$ in $\Omega^+$ and $w = \gamma$ on $\partial\Omega^+$. This completes the proof.

# Chapter 3

# Harmonic Boundary Value Problems for the Poisson Equation

In this Chapter, we study a harmonic Green and a harmonic Neumann functions in the fan-shaped domain $\Omega^+$ with angle $\pi/\alpha$ ($\alpha \geq 1/2$), and then solve the corresponding Dirichlet and Neumann problems for the Poisson equation. Here $\Omega^+,\ \Gamma_0,\ \varpi,\ \vartheta$ are given as in Section 2.4.

## 3.1 Harmonic Dirichlet Problem

**Definition 3.1.1.** [6] *A real-valued function* $G(z,\zeta) = \dfrac{1}{2}G_1(z,\zeta)$ *in a regular domain* $D \subset \mathbb{C}$ *is called the Green function of* $D$*, more exactly the Green function of* $D$ *for the Laplace operator, if for any fixed* $\zeta \in D$ *as a function of* $z$*, it possesses the following properties:*
1. $G(z,\zeta)$ *is harmonic in* $D\backslash\{\zeta\}$*,*
2. $G(z,\zeta) + \log|\zeta - z|$ *is harmonic in* $D$*,*
3. $\lim\limits_{z\to t,\ z\in D} G(z,\zeta) = 0, \quad t \in \partial D.$

The Green function is uniquely determined by 1-3. Actually, it also has the additional properties:
4. $G(z,\zeta) > 0$,
5. $G(z,\zeta) = G(\zeta, z)$.
It must be noted that not any domain in the complex plane has a Green function. The existence of the Green function for a given domain $D \subset \mathbb{C}$ can be proved when the Dirichlet problem for harmonic functions is solvable for $D$.

The harmonic Green function $G_1$ for the fan-shaped domain $\Omega^+$ with angle

$\vartheta = \pi/\alpha$ is

$$G_1(z,\zeta) = \log \left| \frac{\left(\overline{\zeta^\alpha} - z^\alpha\right)\left(1 - z^\alpha \overline{\zeta^\alpha}\right)}{\left(\zeta^\alpha - z^\alpha\right)\left(1 - z^\alpha \zeta^\alpha\right)} \right|^2, \tag{3.1}$$

which can be verified easily.

The outward normal derivative of the boundary $\partial\Omega^+$ is given by

$$\partial_{\nu_z} = \begin{cases} z\partial_z + \bar{z}\partial_{\bar{z}}, & z \in \Gamma_0 \backslash \{1, \varpi\}, \\ -i(\partial_z - \partial_{\bar{z}}), & z \in (0,1), \\ i(e^{i\vartheta}\partial_z - e^{-i\vartheta}\partial_{\bar{z}}), & z \in (\varpi, 0). \end{cases} \tag{3.2}$$

Thus, for $z \in \Gamma_0 \backslash \{1, \varpi\}$,

$$\begin{aligned} \partial_{\nu_z} G_1(z,\zeta) &= (z\partial_z + \bar{z}\partial_{\bar{z}}) G_1(z,\zeta) \\ &= 2\alpha \left[ \frac{z^\alpha}{\zeta^\alpha - z^\alpha} + \frac{\overline{z^\alpha}}{\overline{\zeta^\alpha} - \overline{z^\alpha}} - \frac{z^\alpha}{\overline{\zeta^\alpha} - z^\alpha} - \frac{\overline{z^\alpha}}{\zeta^\alpha - \overline{z^\alpha}} \right], \end{aligned}$$

and for $z \in (0,1)$,

$$\begin{aligned} \partial_{\nu_z} G_1(z,\zeta) &= -i(\partial_z - \partial_{\bar{z}}) G_1(z,\zeta) \\ &= -2i\alpha z^{\alpha-1} \left[ \frac{1}{\zeta^\alpha - z^\alpha} + \frac{\zeta^\alpha}{1 - z^\alpha \zeta^\alpha} - \frac{1}{\overline{\zeta^\alpha} - z^\alpha} - \frac{\overline{\zeta^\alpha}}{1 - z^\alpha \overline{\zeta^\alpha}} \right]. \end{aligned}$$

When $z \in (\varpi, 0)$, i.e. $z = (1-\rho)e^{i\vartheta}, 0 < \rho < 1$, we have $z^\alpha = \overline{z^\alpha}$ and $e^{i\vartheta} z^{\alpha-1} = e^{-i\vartheta}\overline{z^{\alpha-1}}$, then

$$\begin{aligned} \partial_{\nu_z} G_1(z,\zeta) &= i(e^{i\vartheta}\partial_z - e^{-i\vartheta}\partial_{\bar{z}}) G_1(z,\zeta) \\ &= 2i\alpha e^{i\vartheta} z^{\alpha-1} \left[ \frac{1}{\zeta^\alpha - z^\alpha} + \frac{\zeta^\alpha}{1 - z^\alpha \zeta^\alpha} - \frac{1}{\overline{\zeta^\alpha} - z^\alpha} - \frac{\overline{\zeta^\alpha}}{1 - z^\alpha \overline{\zeta^\alpha}} \right]. \end{aligned}$$

From [27], we obtain the following representation formula.

**Theorem 3.1.1.** *Any $w \in C^2(\Omega^+; \mathbb{C}) \cap C^1(\overline{\Omega^+}; \mathbb{C})$ can be represented as*

$$w(z) = -\frac{1}{4\pi} \int_{\partial\Omega^+} w(\zeta) \partial_{\nu_\zeta} G_1(z,\zeta) \mathrm{d}s_\zeta - \frac{1}{\pi} \int_{\Omega^+} w_{\zeta\bar{\zeta}}(\zeta) G_1(z,\zeta) \mathrm{d}\xi \mathrm{d}\eta,$$

*where $s_\zeta$ is the arc length parameter on $\partial\Omega^+$ with respect to the variable $\zeta$ and $G_1(z,\zeta)$ is the harmonic Green function for $\Omega^+$.*

Based on Theorem 3.1.1, the representation formula provides a solution for the related Dirichlet problem.

**Theorem 3.1.2.** *The Dirichlet problem*

$$\partial_z\partial_{\bar z} w = f \quad in \quad \Omega^+, \quad f \in L_p(\Omega^+;\mathbb{C}),\ p>2, \quad w=\gamma \quad on \quad \partial\Omega^+, \quad \gamma \in C(\partial\Omega^+;\mathbb{C})$$

*is uniquely solvable by*

$$\begin{aligned} w(z) \ &= \frac{\alpha}{2\pi i}\int_{\Gamma_0} \gamma(\zeta)\left[\frac{\zeta^\alpha}{\zeta^\alpha - z^\alpha} + \frac{\overline{\zeta^\alpha}}{\overline{\zeta^\alpha}-\overline{z^\alpha}} - \frac{\overline{\zeta^\alpha}}{\overline{\zeta^\alpha}-z^\alpha} - \frac{\zeta^\alpha}{\zeta^\alpha-\overline{z^\alpha}}\right]\frac{\mathrm{d}\zeta}{\zeta} \\ &+\frac{\alpha}{2\pi i}\int_0^1 \gamma(\zeta)\left[\frac{1}{\zeta^\alpha - z^\alpha} - \frac{1}{\zeta^\alpha-\overline{z^\alpha}} - \frac{z^\alpha}{1-z^\alpha\zeta^\alpha} + \frac{\overline{z^\alpha}}{1-\overline{z^\alpha}\zeta^\alpha}\right]\zeta^{\alpha-1}\mathrm{d}\zeta \\ &+\frac{\alpha}{2\pi i}\int_{[\varpi,0]} \gamma(\zeta)\left[\frac{1}{\zeta^\alpha - z^\alpha} - \frac{1}{\zeta^\alpha-\overline{z^\alpha}} - \frac{z^\alpha}{1-z^\alpha\zeta^\alpha} + \frac{\overline{z^\alpha}}{1-\overline{z^\alpha}\zeta^\alpha}\right]\zeta^{\alpha-1}\mathrm{d}\zeta \\ &-\frac{1}{\pi}\int_{\Omega^+} f(\zeta)G_1(z,\zeta)\mathrm{d}\xi\mathrm{d}\eta, \quad z\in\Omega, \end{aligned} \tag{3.3}$$

*where $G_1$ is given in* (3.1)*.*

**Proof:** Obviously, if the problem is solvable, it must be unique. By Theorem 3.1.1, we only need to verify that (3.3) is a solution. Since

$$\partial_z G_1(z,\zeta) = \alpha z^{\alpha-1}\left[\frac{1}{\zeta^\alpha - z^\alpha} + \frac{\zeta^\alpha}{1-z^\alpha\zeta^\alpha} - \frac{1}{\overline{\zeta^\alpha}-z^\alpha} - \frac{\overline{\zeta^\alpha}}{1-z^\alpha\overline{\zeta^\alpha}}\right],$$

then similarly to the proof of Theorem 2.4.2, $\dfrac{\alpha z^{\alpha-1}}{\zeta^\alpha - z^\alpha}$ can be written as

$$\frac{\alpha z^{\alpha-1}}{\zeta^\alpha - z^\alpha} = \frac{1}{\zeta - z} + \widetilde{g}(z,\zeta),$$

where for any $z\in\Omega^+$, $\widetilde{g}(z,\zeta)$ is analytic in $z$. Thus, we obviously see that expression (3.3) is a solution to the Poisson equation. Moreover, by the properties of the Green function, the area integral in (3.3) vanishes on $\partial\Omega^+$. Actually, we can rewrite $w(z)$ as

$$w(z) = \frac{\alpha}{2\pi i}\int_{\partial\Omega^+} \gamma(\zeta)H(z,\zeta)\mathrm{d}\zeta - \frac{1}{\pi}\int_{\Omega^+} f(\zeta)G_1(z,\zeta)\mathrm{d}\xi\mathrm{d}\eta,$$

with $H(z,\zeta)$ defined by (2.80). Then, similarly from the proof of Theorem 2.4.2, we obtain

$$\lim_{z\to\zeta} w(z) = \gamma(\zeta) \quad \text{for} \quad \zeta \in \partial\Omega^+.$$

This completes the proof.

## 3.2 Harmonic Neumann Problem

**Definition 3.2.1.** [6] *A real-valued function* $N(z,\zeta) = \frac{1}{2}N_1(z,\zeta)$ *in a regular domain* $D$ *is called a Neumann function (for the Laplace operator), if as a function of* $z$*, it satisfies*

1. $N(z,\zeta)$ *is harmonic in* $D\backslash\{\zeta\}$*,*
2. $N(z,\zeta) + \log|\zeta - z|$ *is harmonic in* $D$*,*
3. $\partial_{\nu_z} N(z,\zeta)$ *is constant on any boundary component of* $D$ *for any* $\zeta \in D$*.*

**Remark 3.2.1.** The Neumann function is not uniquely defined by 1-3 above. It is only given up to an arbitrary additive constant.

**Example 3.2.1.** A harmonic Neumann function for the ring domain $R = \{z \in \mathbb{C} :\ 0 < r < |z| < 1\}$ can be written as (see [54])

$$\widehat{N_1}(z,\zeta) = -\log\left|(\zeta - z)(1 - z\overline{\zeta}) \times \prod_{k=1}^{\infty} \frac{(z - r^{2k}\zeta)(z\overline{\zeta} - r^{2k})(\zeta - r^{2k}z)(1 - r^{2k}z\overline{\zeta})}{|z|^2|\zeta|^2}\right|^2,$$

and its boundary behavior is

$$\partial_{\nu_z}\widehat{N_1}(z,\zeta) = \begin{cases} -2, & |z| = 1, \\ 0, & |z| = r. \end{cases}$$

A harmonic Neumann function $N_1(z,\zeta)$ for the Poisson equation in $\Omega^+$ can be expressed as

$$N_1(z,\zeta) = -\log\left|\left(\zeta^\alpha - z^\alpha\right)\left(\overline{\zeta^\alpha} - z^\alpha\right)\left(1 - z^\alpha\zeta^\alpha\right)\left(1 - z^\alpha\overline{\zeta^\alpha}\right)\right|^2. \tag{3.4}$$

We verify this in the following. For $z \in \Gamma_0 \backslash \{1, \varpi\}$,

$$N_1(z,\zeta) = -2\log|\big(\zeta^\alpha - z^\alpha\big)\big(\overline{\zeta^\alpha} - z^\alpha\big)|^2,$$

$$\begin{aligned}\partial_{\nu_z} N_1(z,\zeta) &= (z\partial_z + \bar{z}\partial_{\bar{z}})N_1(z,\zeta)\\ &= \alpha z^\alpha\left[\frac{1}{\zeta^\alpha - z^\alpha} + \frac{1}{\overline{\zeta^\alpha} - z^\alpha} + \frac{\zeta^\alpha}{1 - z^\alpha\zeta^\alpha} + \frac{\overline{\zeta^\alpha}}{1 - z^\alpha\overline{\zeta^\alpha}}\right]\\ &\quad + \alpha\overline{z^\alpha}\left[\frac{1}{\overline{\zeta^\alpha} - \overline{z^\alpha}} + \frac{1}{\zeta^\alpha - \overline{z^\alpha}} + \frac{\zeta^\alpha}{1 - \overline{z^\alpha}\zeta^\alpha} + \frac{\overline{\zeta^\alpha}}{1 - \overline{z^\alpha}\overline{\zeta^\alpha}}\right] = -4\alpha,\end{aligned}$$

when $z \in (0,1)$,

$$N_1(z,\zeta) = -2\log|\big(\zeta^\alpha - z^\alpha\big)\big(1 - z^\alpha\zeta^\alpha\big)|^2,$$

$$\begin{aligned}\partial_{\nu_z} N_1(z,\zeta) &= -i(\partial_z - \partial_{\bar{z}})N_1(z,\zeta) = -i\alpha z^{\alpha-1}\Big[\frac{1}{\zeta^\alpha - z^\alpha} + \frac{1}{\overline{\zeta^\alpha} - z^\alpha} + \frac{\zeta^\alpha}{1 - z^\alpha\zeta^\alpha}\\ &\quad + \frac{\overline{\zeta^\alpha}}{1 - z^\alpha\overline{\zeta^\alpha}} - \frac{1}{\overline{\zeta^\alpha} - \overline{z^\alpha}} - \frac{1}{\zeta^\alpha - \overline{z^\alpha}} - \frac{\zeta^\alpha}{1 - \overline{z^\alpha}\zeta^\alpha} - \frac{\overline{\zeta^\alpha}}{1 - \overline{z^\alpha}\overline{\zeta^\alpha}}\Big] = 0,\end{aligned}$$

and for $z \in (\varpi, 0)$, $z = (1-\rho)e^{i\vartheta}, \quad 0 < \rho < 1$,

$$N_1(z,\zeta) = -2\log|\big(\zeta^\alpha - z^\alpha\big)\big(1 - z^\alpha\zeta^\alpha\big)|^2,$$

$$\begin{aligned}&\partial_{\nu_z} N_1(z,\zeta) = i(e^{i\vartheta}\partial_z - e^{-i\vartheta}\partial_{\bar{z}})N_1(z,\zeta)\\ &= i\alpha z^{\alpha-1}e^{i\vartheta}\Big[\frac{1}{\zeta^\alpha - z^\alpha} + \frac{1}{\overline{\zeta^\alpha} - z^\alpha} + \frac{\zeta^\alpha}{1 - z^\alpha\zeta^\alpha} + \frac{\overline{\zeta^\alpha}}{1 - z^\alpha\overline{\zeta^\alpha}}\Big]\\ &\quad - i\alpha\overline{z^{\alpha-1}}e^{-i\vartheta}\Big[\frac{1}{\overline{\zeta^\alpha} - \overline{z^\alpha}} + \frac{1}{\zeta^\alpha - \overline{z^\alpha}} + \frac{\zeta^\alpha}{1 - \overline{z^\alpha}\zeta^\alpha} + \frac{\overline{\zeta^\alpha}}{1 - \overline{z^\alpha}\overline{\zeta^\alpha}}\Big] = 0.\end{aligned}$$

Moreover, the normalization condition holds, that is,

$$\int_{\Gamma_0} N_1(z,\zeta)\frac{\mathrm{d}z}{z} = -\frac{2}{\alpha}\int_{|z|=1} \log|z - \zeta^\alpha|^2\frac{\mathrm{d}z}{z} = 0.$$

Thus, $N_1(z,\zeta)$ satisfies the properties:

1. $N_1(z,\zeta)$ is harmonic in $z \in \Omega^+ \backslash \{\zeta\}$,
2. $N_1(z,\zeta) + \log|\zeta - z|^2$ is harmonic in $z \in \Omega^+$ for any $\zeta \in \overline{\Omega^+}$,
3. $\partial_{\nu_z} N_1(z,\zeta) = \begin{cases} -4\alpha, & z \in \Gamma_0 \backslash \{1,\varpi\}, \\ 0, & z \in (0,1) \cup (\varpi, 0), \end{cases} \quad \text{for } \zeta \in \Omega^+,$

4. $\displaystyle\int_{\Gamma_0} N_1(z,\zeta)\frac{\mathrm{d}z}{z} = 0.$

Then, we give a representation formula for the domain $\Omega^+$.

**Theorem 3.2.1.** *Any $w \in C^2(\Omega^+;\mathbb{C}) \cap C^1(\overline{\Omega^+};\mathbb{C})$ can be represented as*

$$w(z) = \frac{\alpha}{\pi i}\int_{\Gamma_0} w(\zeta)\frac{\mathrm{d}\zeta}{\zeta} + \frac{1}{4\pi}\int_{\partial\Omega^+} \partial_{\nu_\zeta} w(\zeta) N_1(z,\zeta)\mathrm{d}s_\zeta - \frac{1}{\pi}\int_{\Omega^+} w_{\zeta\overline{\zeta}}(\zeta)N_1(z,\zeta)\mathrm{d}\xi\mathrm{d}\eta.$$

*where $N_1(z,\zeta)$ is the harmonic Neumann function for $\Omega^+$.*

**Proof:** Let $z \in \Omega^+$ and $\varepsilon > 0$ such that

$$\overline{B_\varepsilon(z)} \subset \Omega^+ \ \text{ with } \ B_\varepsilon(z) = \{\zeta \in \mathbb{C} : \ |\zeta - z| < \varepsilon\}.$$

Suppose that $\Omega_\varepsilon = \Omega^+ \backslash \overline{B_\varepsilon(z)}$, then

$$\begin{aligned}
&\frac{1}{\pi}\int_{\Omega_\varepsilon} w_{\zeta\overline{\zeta}}(\zeta)N_1(z,\zeta)\mathrm{d}\xi\mathrm{d}\eta = \frac{1}{2\pi}\int_{\Omega_\varepsilon}\Big\{\partial_{\overline{\zeta}}[w_\zeta(\zeta)N_1(z,\zeta)] + \partial_\zeta[w_{\overline{\zeta}}(\zeta)N_1(z,\zeta)] \\
&\qquad\qquad - w_\zeta(\zeta)N_{1\overline{\zeta}}(z,\zeta) - w_{\overline{\zeta}}(\zeta)N_{1\zeta}(z,\zeta)\Big\}\mathrm{d}\xi\mathrm{d}\eta \\
&= \frac{1}{4\pi i}\int_{\partial\Omega_\varepsilon} N_1(z,\zeta)[w_\zeta(\zeta)\mathrm{d}\zeta - w_{\overline{\zeta}}(\zeta)\mathrm{d}\overline{\zeta}] \\
&\quad - \frac{1}{2\pi}\int_{\Omega_\varepsilon}\Big\{\partial_\zeta[w(\zeta)N_{1\overline{\zeta}}(z,\zeta)] + \partial_{\overline{\zeta}}[w(\zeta)N_{1\zeta}(z,\zeta)] - 2\partial_\zeta\partial_{\overline{\zeta}}N_1(z,\zeta)w(\zeta)\Big\}\mathrm{d}\xi\mathrm{d}\eta \\
&= \frac{1}{4\pi i}\int_{\partial\Omega_\varepsilon} N_1(z,\zeta)[w_\zeta(\zeta)\mathrm{d}\zeta - w_{\overline{\zeta}}(\zeta)\mathrm{d}\overline{\zeta}] - \frac{1}{4\pi i}\int_{\partial\Omega_\varepsilon} w(\zeta)[N_{1\zeta}(z,\zeta)\mathrm{d}\zeta - N_{1\overline{\zeta}}(z,\zeta)]\mathrm{d}\overline{\zeta}] \\
&= \frac{1}{4\pi}\int_{\partial\Omega^+}\big[\partial_{\nu_\zeta}w(\zeta)N_1(z,\zeta) - w(\zeta)\partial_{\nu_\zeta}N_1(z,\zeta)\big]\mathrm{d}s_\zeta \\
&\quad - \frac{1}{4\pi i}\int_{|\zeta-z|=\varepsilon} N_1(z,\zeta)\Big[(\zeta-z)w_\zeta(\zeta) + \overline{(\zeta-z)}w_{\overline{\zeta}}(\zeta)\Big]\frac{\mathrm{d}\zeta}{\zeta-z} \\
&\quad + \frac{1}{4\pi i}\int_{|\zeta-z|=\varepsilon}\Big[(\zeta-z)N_{1\zeta}(z,\zeta) + \overline{(\zeta-z)}N_{1\overline{\zeta}}(z,\zeta)\Big]w(\zeta)\frac{\mathrm{d}\zeta}{\zeta-z}.
\end{aligned}$$

Introducing the polar coordinates $\zeta = z + \varepsilon e^{i\psi}$ gives rise to

$$
\begin{aligned}
&-\frac{1}{4\pi i} \int\limits_{|\zeta - z| = \varepsilon} N_1(z,\zeta) \left[(\zeta - z)w_\zeta(\zeta) + \overline{(\zeta - z)}w_{\overline{\zeta}}(\zeta)\right] \frac{\mathrm{d}\zeta}{\zeta - z} \\
&= -\frac{\varepsilon}{4\pi} \int\limits_0^{2\pi} N_1(z,\zeta)[e^{i\psi}w_\zeta(\zeta) + e^{-i\psi}w_{\overline{\zeta}}(\zeta)]\mathrm{d}\psi,
\end{aligned}
$$

which tends to 0 when $\varepsilon \to 0$. Similarly,

$$
\begin{aligned}
&\lim_{\varepsilon \to 0} \frac{1}{4\pi i} \int\limits_{|\zeta - z| = \varepsilon} \left[(\zeta - z)N_{1\zeta}(z,\zeta) + \overline{(\zeta - z)}N_{1\overline{\zeta}}(z,\zeta)\right] w(\zeta) \frac{\mathrm{d}\zeta}{\zeta - z} \\
&= -\lim_{\varepsilon \to 0} \frac{1}{2\pi} \int\limits_0^{2\pi} w(z + \varepsilon e^{i\varphi})\mathrm{d}\varphi = -w(z).
\end{aligned}
$$

Then,

$$
w(z) = \frac{1}{4\pi} \int\limits_{\partial\Omega^+} \left[\partial_{\nu_\zeta} w(\zeta)N_1(z,\zeta) - \partial_{\nu_\zeta} N_1(\zeta)w(\zeta)\right] \mathrm{d}s_\zeta - \frac{1}{\pi} \int\limits_{\Omega^+} w_{\zeta\overline{\zeta}}(\zeta)N_1(z,\zeta)\mathrm{d}\xi\mathrm{d}\eta. \tag{3.5}
$$

Hence, the desired result follows from (3.5) and the property 3 of $N_1(z,\zeta)$.

Next, the outward normal derivatives at the corner points are introduced. Let the partial outward normal derivatives be

$$
\partial^+_{\nu_z} w(0) = \lim_{t \to 0,\ t \in (0,1)} \partial_{\nu_z} w(t), \quad \partial^-_{\nu_z} w(0) = \lim_{t \to 0,\ t \in (\varpi,0)} \partial_{\nu_z} w(t), \tag{3.6}
$$

$$
\partial^+_{\nu_z} w(1) = \lim_{t \to 1,\ t \in \Gamma_0 \setminus \{1\}} \partial_{\nu_z} w(t), \quad \partial^-_{\nu_z} w(1) = \lim_{t \to 1,\ t \in (0,1)} \partial_{\nu_z} w(t), \tag{3.7}
$$

and

$$
\partial^+_{\nu_z} w(\varpi) = \lim_{t \to \varpi,\ t \in (\varpi,0)} \partial_{\nu_z} w(t), \quad \partial^-_{\nu_z} w(\varpi) = \lim_{t \to \varpi,\ t \in \Gamma_0 \setminus \{\varpi\}} \partial_{\nu_z} w(t). \tag{3.8}
$$

**Definition 3.2.2.** *If the partial outward normal derivatives* (3.6)-(3.8) *exist, then the outward normal derivatives at the three corner points are,*

$$
\partial_{\nu_z} w(t) = \frac{1}{2}\left[\partial^+_{\nu_z} w(t) + \partial^-_{\nu_z} w(t)\right], \quad t \in \{0,\ 1,\ \varpi\}.
$$

Finally, the Neumann problem for the Poisson equation is discussed.

**Theorem 3.2.2.** The Neumann boundary value problem

$$\partial_z\partial_{\overline{z}}w = f \quad in \quad \Omega^+, \qquad \partial_\nu w = \gamma \quad on \quad \partial\Omega^+, \qquad \frac{\alpha}{\pi}\int_0^{\pi/\alpha} w(e^{i\theta})\mathrm{d}\theta = c_0$$

*for* $f \in L_p(\Omega^+;\mathbb{C}),\ p > 2,\ \gamma \in C(\partial\Omega^+;\mathbb{C})$ *is solvable if and only if*

$$\frac{1}{2\pi}\int\limits_{\partial\Omega^+} \gamma(\zeta)\mathrm{d}s_\zeta = \frac{2}{\pi}\int\limits_{\Omega^+} f(\zeta)\mathrm{d}\xi\mathrm{d}\eta. \tag{3.9}$$

*Then the solution is uniquely expressed by*

$$\begin{aligned} w(z) \ &= c_0 - \frac{1}{2\pi i}\int\limits_{\Gamma_0} \gamma(\zeta)\log|\big(\zeta^\alpha - z^\alpha\big)\big(\overline{\zeta^\alpha} - z^\alpha\big)|^2\frac{\mathrm{d}\zeta}{\zeta} \\ &-\frac{1}{2\pi}\int_0^1 \gamma(\zeta)\log|\big(\zeta^\alpha - z^\alpha\big)\big(1 - z^\alpha\zeta^\alpha\big)|^2\mathrm{d}\zeta \\ &+\frac{1}{2\pi}\int\limits_{[\varpi,0]} e^{-i\vartheta}\gamma(\zeta)\log|\big(\zeta^\alpha - z^\alpha\big)\big(1 - z^\alpha\zeta^\alpha\big)|^2\mathrm{d}\zeta \\ &-\frac{1}{\pi}\int\limits_{\Omega^+} f(\zeta)N_1(z,\zeta)\mathrm{d}\xi\mathrm{d}\eta, \quad z \in \Omega, \end{aligned} \tag{3.10}$$

*where* $N_1(z,\zeta)$ *is given in* (3.4).

**Proof:** From Theorem 3.2.1, if the Neumann problem is solvable, it should be of the form (3.10). Then we confirm that (3.10) is a solution. Similarly to the proof of Theorem 3.1.2, we easily get $w_{z\overline{z}} = f$ in $\Omega^+$. Since when $\zeta \in \partial\Omega^+$,

$$\frac{\alpha}{\pi i}\int\limits_{\Gamma_0} \log|(\zeta^\alpha - z^\alpha)(1 - z^\alpha\zeta^\alpha)|^2\frac{\mathrm{d}z}{z} = \frac{1}{\pi i}\int\limits_{|z|=1} \log|1 - z\zeta^\alpha|^2\frac{\mathrm{d}z}{z} = 0, \tag{3.11}$$

thus, the normalization condition

$$\frac{\alpha}{\pi}\int_0^{\pi/\alpha} w(e^{i\theta})\mathrm{d}\theta = c_0$$

follows from (3.11) and the normalization of $N_1(z,\zeta)$.

In addition, we have

$$\begin{aligned}
&(z\partial_z+\bar z\partial_{\bar z})w(z)\\
&=\frac{\alpha}{2\pi i}\int\limits_{\Gamma_0}\gamma(\zeta)\left[\frac{\zeta^\alpha}{\zeta^\alpha-z^\alpha}+\frac{\overline{\zeta^\alpha}}{\overline{\zeta^\alpha}-\overline{z^\alpha}}+\frac{z^\alpha}{\overline{\zeta^\alpha}-z^\alpha}+\frac{\overline{z^\alpha}}{\zeta^\alpha-\overline{z^\alpha}}-2\right]\frac{\mathrm{d}\zeta}{\zeta}\\
&\quad+\frac{\alpha}{2\pi}\int\limits_0^1\gamma(\zeta)\left[\frac{z^\alpha}{\zeta^\alpha-z^\alpha}+\frac{z^\alpha\zeta^\alpha}{1-z^\alpha\zeta^\alpha}+\frac{\overline{z^\alpha}}{\zeta^\alpha-\overline{z^\alpha}}+\frac{\overline{z^\alpha}\zeta^\alpha}{1-\overline{z^\alpha}\zeta^\alpha}\right]\mathrm{d}\zeta\\
&\quad-\frac{\alpha}{2\pi}\int\limits_{[\varpi,0]}e^{-i\vartheta}\gamma(\zeta)\left[\frac{z^\alpha}{\zeta^\alpha-z^\alpha}+\frac{z^\alpha\zeta^\alpha}{1-z^\alpha\zeta^\alpha}+\frac{\overline{z^\alpha}}{\zeta^\alpha-\overline{z^\alpha}}+\frac{\overline{z^\alpha}\zeta^\alpha}{1-\overline{z^\alpha}\zeta^\alpha}\right]\mathrm{d}\zeta\\
&\quad-\frac{1}{\pi}\int\limits_{\Omega^+}f(\zeta)(z\partial_z+\bar z\partial_{\bar z})N_1(z,\zeta)\mathrm{d}\xi\mathrm{d}\eta.
\end{aligned}$$

Hence, for $\zeta_0\in\Gamma_0\backslash\{1,\ \varpi\}$,

$$\begin{aligned}
\partial_{\nu_z}w(\zeta_0)&=\lim_{z\in\Omega^+,z\to\zeta_0}(z\partial_z+\bar z\partial_{\bar z})w(z)\\
&=\lim_{z\in\Omega^+,z\to\zeta_0}\left\{\frac{\alpha}{2\pi i}\int\limits_{\Gamma_0}\gamma(\zeta)\left[\frac{\zeta^\alpha}{\zeta^\alpha-z^\alpha}+\frac{\overline{\zeta^\alpha}}{\overline{\zeta^\alpha}-\overline{z^\alpha}}-1\right]\frac{\mathrm{d}\zeta}{\zeta}\right.\\
&\qquad\left.-\frac{\alpha}{\pi}\int\limits_{\partial\Omega^+}\gamma(\zeta)\mathrm{d}s_\zeta+\frac{4\alpha}{\pi}\int\limits_{\Omega^+}f(\zeta)\mathrm{d}\xi\mathrm{d}\eta\right\}\\
&=\gamma(\zeta_0)-\frac{\alpha}{\pi}\int\limits_{\partial\Omega^+}\gamma(\zeta)\mathrm{d}s_\zeta+\frac{4\alpha}{\pi}\int\limits_{\Omega^+}f(\zeta)\mathrm{d}\xi\mathrm{d}\eta,
\end{aligned}$$

which implies the sufficiency and necessary of (3.9). Further, from $\gamma\in C(\partial\Omega^+,\mathbb{C})$,

$$\partial^+_{\nu_z}w(1)=\lim_{\zeta_0\to1,\ \zeta_0\in\Gamma_0\backslash\{1,\varpi\}}\partial_{\nu_z}w(\zeta_0)=\gamma(1),\tag{3.12}$$

$$\partial^-_{\nu_z}w(\varpi)=\lim_{\zeta_0\to\varpi,\ \zeta_0\in\Gamma_0\backslash\{1,\varpi\}}\partial_{\nu_z}w(\zeta_0)=\gamma(\varpi).\tag{3.13}$$

Similarly,

$$\begin{aligned}
&-i(\partial_z-\partial_{\bar z})w(z)\\
&=-\frac{\alpha}{2\pi}\int\limits_{\Gamma_0}\gamma(\zeta)\left[\frac{z^{\alpha-1}}{\zeta^\alpha-z^\alpha}+\frac{z^{\alpha-1}}{\overline{\zeta^\alpha}-z^\alpha}-\frac{\overline{z^{\alpha-1}}}{\zeta^\alpha-\overline{z^\alpha}}-\frac{\overline{z^{\alpha-1}}}{\overline{\zeta^\alpha}-\overline{z^\alpha}}\right]\frac{\mathrm{d}\zeta}{\zeta}\\
&\quad+\frac{\alpha}{2\pi i}\int\limits_0^1\gamma(\zeta)\left[\frac{z^{\alpha-1}}{\zeta^\alpha-z^\alpha}+\frac{z^{\alpha-1}\zeta^\alpha}{1-z^\alpha\zeta^\alpha}-\frac{\overline{z^{\alpha-1}}}{\zeta^\alpha-\overline{z^\alpha}}-\frac{\overline{z^{\alpha-1}}\zeta^\alpha}{1-\overline{z^\alpha}\zeta^\alpha}\right]\mathrm{d}\zeta
\end{aligned}$$

$$-\frac{\alpha}{2\pi i}\int\limits_{[\varpi,0]} e^{-i\vartheta}\gamma(\zeta)\left[\frac{z^{\alpha-1}}{\zeta^\alpha-z^\alpha}+\frac{z^{\alpha-1}\zeta^\alpha}{1-z^\alpha\zeta^\alpha}-\frac{\overline{z^{\alpha-1}}}{\zeta^\alpha-\overline{z^\alpha}}-\frac{\overline{z^{\alpha-1}}\zeta^\alpha}{1-\overline{z^\alpha}\zeta^\alpha}\right]\mathrm{d}\zeta$$
$$-\frac{1}{\pi i}\int\limits_{\Omega^+} f(\zeta)(\partial_z-\partial_{\bar z})N_1(z,\zeta)\mathrm{d}\xi\mathrm{d}\eta.$$

Then for $\zeta_0\in(0,1)$,

$$\begin{aligned}\partial_{\nu_z}w(\zeta_0) &= \lim_{z\in\Omega^+,z\to\zeta_0} -i(\partial_z-\partial_{\bar z})w(z)\\ &= \lim_{z\to\Omega^+,z\to\zeta_0}\frac{\alpha}{2\pi i}\int_0^1\gamma(\zeta)\left[\frac{z^{\alpha-1}}{\zeta^\alpha-z^\alpha}-\frac{\overline{z^{\alpha-1}}}{\zeta^\alpha-\overline{z^\alpha}}\right]\mathrm{d}\zeta\\ &= \lim_{\substack{z\in\Omega^+\\ z\to\zeta_0}}\left\{\frac{\alpha}{2\pi i}\int_0^1\gamma(\zeta)\left[\frac{z^{\alpha-1}-\zeta^{\alpha-1}}{\zeta^\alpha-z^\alpha}-\frac{\overline{z^{\alpha-1}}-\zeta^{\alpha-1}}{\zeta^\alpha-\overline{z^\alpha}}\right]\mathrm{d}\zeta\right.\\ &\qquad\left.+\frac{\alpha}{2\pi i}\int_0^1\gamma(\zeta)\left[\frac{1}{\zeta^\alpha-z^\alpha}-\frac{1}{\zeta^\alpha-\overline{z^\alpha}}\right]\zeta^{\alpha-1}\mathrm{d}\zeta\right\}\\ &= \lim_{z\in\Omega^+,z\to\zeta_0}\frac{1}{2\pi i}\int_0^1\gamma(\zeta^{1/\alpha})\left[\frac{1}{\zeta-z^\alpha}-\frac{1}{\zeta-\overline{z^\alpha}}\right]\mathrm{d}\zeta=\gamma(\zeta_0).\end{aligned}$$

Thus,

$$\partial^+_{\nu_z}w(0)=\lim_{\zeta_0\to0,\ \zeta_0\in(0,1)}\partial_{\nu_z}w(\zeta_0)=\gamma(0), \tag{3.14}$$

$$\partial^-_{\nu_z}w(1)=\lim_{\zeta_0\to1,\ \zeta_0\in(0,1)}\partial_{\nu_z}w(\zeta_0)=\gamma(1). \tag{3.15}$$

Finally, we also observe that,

$$\begin{aligned}&i(e^{i\vartheta}\partial_z-e^{-i\vartheta}\partial_{\bar z})w(z)\\ &=\frac{\alpha}{2\pi}\int\limits_{\Gamma_0}\gamma(\zeta)\left[\frac{e^{i\vartheta}z^{\alpha-1}}{\zeta^\alpha-z^\alpha}+\frac{e^{i\vartheta}z^{\alpha-1}}{\overline{\zeta^\alpha}-z^\alpha}-\frac{e^{-i\vartheta}\overline{z^{\alpha-1}}}{\zeta^\alpha-\overline{z^\alpha}}-\frac{e^{-i\vartheta}\overline{z^{\alpha-1}}}{\overline{\zeta^\alpha}-\overline{z^\alpha}}\right]\frac{\mathrm{d}\zeta}{\zeta}\\ &\quad-\frac{\alpha}{2\pi i}\int_0^1\gamma(\zeta)\left[\frac{e^{i\vartheta}z^{\alpha-1}}{\zeta^\alpha-z^\alpha}+\frac{e^{i\vartheta}z^{\alpha-1}\zeta^\alpha}{1-z^\alpha\zeta^\alpha}-\frac{e^{-i\vartheta}\overline{z^{\alpha-1}}}{\zeta^\alpha-\overline{z^\alpha}}-\frac{e^{-i\vartheta}\overline{z^{\alpha-1}}\zeta^\alpha}{1-\overline{z^\alpha}\zeta^\alpha}\right]\mathrm{d}\zeta\\ &\quad+\frac{\alpha}{2\pi i}\int\limits_{[\varpi,0]}e^{-i\vartheta}\gamma(\zeta)\left[\frac{e^{i\vartheta}z^{\alpha-1}}{\zeta^\alpha-z^\alpha}+\frac{e^{i\vartheta}z^{\alpha-1}\zeta^\alpha}{1-z^\alpha\zeta^\alpha}-\frac{e^{-i\vartheta}\overline{z^{\alpha-1}}}{\zeta^\alpha-\overline{z^\alpha}}-\frac{e^{-i\vartheta}\overline{z^{\alpha-1}}\zeta^\alpha}{1-\overline{z^\alpha}\zeta^\alpha}\right]\mathrm{d}\zeta\\ &\quad+\frac{1}{\pi i}\int\limits_{\Omega^+}f(\zeta)(e^{i\vartheta}\partial_z-e^{-i\vartheta}\partial_{\bar z})N_1(z,\zeta)\mathrm{d}\xi\mathrm{d}\eta.\end{aligned}$$

Since for $\zeta_0 \in (\varpi, 0)$, that is, $\zeta_0 = (1-\rho)e^{i\vartheta}$, $0 < \rho < 1$, we obtain

$$\zeta_0^\alpha = \overline{\zeta_0^\alpha}, \quad e^{i\vartheta}{\zeta_0}^{\alpha-1} = e^{-i\vartheta}\overline{\zeta_0^{\alpha-1}}.$$

Thus,

$$\begin{aligned}
\partial_{\nu_z} w(\zeta_0) &= \lim_{z\in\Omega^+, z\to\zeta_0} i(e^{i\vartheta}\partial_z - e^{-i\vartheta}\partial_{\overline{z}})w(z) \\
&= \lim_{z\in\Omega^+, z\to\zeta_0} \frac{\alpha}{2\pi i} \int\limits_{[\varpi,0]} e^{-i\vartheta}\gamma(\zeta)\left[\frac{e^{i\vartheta}z^{\alpha-1}}{\zeta^\alpha - z^\alpha} - \frac{e^{-i\vartheta}\overline{z^{\alpha-1}}}{\zeta^\alpha - \overline{z^\alpha}}\right] \mathrm{d}\zeta \\
&= \lim_{z\in\Omega^+, z\to\zeta_0} \left\{ \frac{\alpha}{2\pi i} \int\limits_{[\varpi,0]} e^{-i\vartheta}\gamma(\zeta)\left[\frac{e^{i\vartheta}(z^{\alpha-1} - \zeta^{\alpha-1})}{\zeta^\alpha - z^\alpha} - \frac{e^{-i\vartheta}(\overline{z^{\alpha-1}} - \overline{\zeta^{\alpha-1}})}{\zeta^\alpha - \overline{z^\alpha}}\right] \mathrm{d}\zeta \right. \\
&\quad \left. + \frac{\alpha}{2\pi i} \int\limits_{[\varpi,0]} e^{-i\vartheta}\gamma(\zeta)\left[\frac{e^{i\vartheta}\zeta^{\alpha-1}}{\zeta^\alpha - z^\alpha} - \frac{e^{-i\vartheta}\overline{\zeta^{\alpha-1}}}{\zeta^\alpha - \overline{z^\alpha}}\right] \mathrm{d}\zeta \right\} \\
&= \lim_{z\in\Omega^+, z\to\zeta_0} \frac{\alpha}{2\pi i} \int\limits_{[\varpi,0]} \gamma(\zeta)\left[\frac{1}{\zeta^\alpha - z^\alpha} - \frac{1}{\zeta^\alpha - \overline{z^\alpha}}\right] \zeta^{\alpha-1}\mathrm{d}\zeta = \gamma(\zeta_0).
\end{aligned}$$

Furthermore,

$$\partial^-_{\nu_z} w(0) = \lim_{\zeta_0\to 0,\ \zeta_0\in(\varpi,0)} \partial_{\nu_z} w(\zeta_0) = \gamma(0), \tag{3.16}$$

$$\partial^+_{\nu_z} w(\varpi) = \lim_{\zeta_0\to \varpi,\ \zeta_0\in(\varpi,0)} \partial_{\nu_z} w(\zeta_0) = \gamma(\varpi). \tag{3.17}$$

Hence, by (3.12)-(3.17) and Definition 3.2.2,

$$\partial_{\nu_z} w(t) = \gamma(t), \quad t \in \{0,\ 1,\ \varpi\}.$$

Then the proof is completed.

# Chapter 4

# Boundary Value Problems for the Bi-Poisson Equation

As stated before, convoluting the harmonic Green function with itself leads to a biharmonic Green function. Similarly, convoluting the harmonic Neumann function with itself also gives rise to a biharmonic Neumann function. This chapter is devoted to the construction of a biharmonic Green function and a biharmonic Neumann function in the fan-shaped domain $\Omega$ with angle $\pi/n$ ($n \in \mathbb{N}$), by using the concept of convolution and some proper transformations, and finally give the solutions and solvability conditions for the corresponding Dirichlet and Neumann problems explicitly. Here $\Omega,\ L,\ \omega = e^{i\theta},\ \theta$ are given as in Section 2.2.

## 4.1 Biharmonic Green Function

In the last Chapter, we have studied the harmonic Green and the harmonic Neumann function explicitly in the fan-shaped domain $\Omega^+$ with angle $\pi/\alpha$ ($\alpha \geq 1/2$). Thus, the harmonic Green function for $\Omega$ is expressed as

$$\begin{aligned} G_1(z,\zeta) &= \log\left|\frac{(\overline{\zeta^n} - z^n)(1 - z^n\overline{\zeta^n})}{(\zeta^n - z^n)(1 - z^n\zeta^n)}\right|^2 \\ &= \sum_{k=0}^{n-1} \log\left|\frac{(\omega^{-2k} - z\overline{\zeta})(\overline{\zeta} - z\omega^{2k})}{(\omega^{-2k} - z\zeta)(\zeta - z\omega^{2k})}\right|^2, \qquad z,\ \zeta \in \Omega. \end{aligned} \tag{4.1}$$

**Remark 4.1.1.** Here the factorization is used. We only take $\zeta^n - z^n$ as an example. For $k = 0, 1, \cdots, n-1$ and $\theta = \pi/n$, we have $0 \leq 2k\theta < 2\pi$. Then $\zeta = z\omega^{2k}$ are $n$ different roots of $\zeta^n - z^n$ in the complex plane. Hence,

$$|\zeta^n - z^n|^2 = \left|\prod_{k=0}^{n-1}(\zeta - z\omega^{2k})\right|^2.$$

What is more, the solution to the Dirichlet problem for $\Omega$ $(\alpha = n)$ is given as follows.

**Lemma 4.1.1.** *The Dirichlet problem for the Poisson equation*

$$w_{z\bar{z}} = f \quad in \quad \Omega, \quad f \in L_p(\Omega;\mathbb{C}),\ p > 2, \qquad w = \gamma \quad on \quad \partial\Omega, \quad \gamma \in C(\partial\Omega;\mathbb{C})$$

*is uniquely solvable by*

$$\begin{aligned} w(z) \quad &= \frac{1}{2\pi i}\int\limits_{L} \gamma(\zeta)\sum_{k=0}^{n-1}\left[\frac{\zeta}{\zeta - z\omega^{2k}} + \frac{\overline{\zeta}}{\overline{\zeta} - \overline{z}\omega^{-2k}} - \frac{\overline{\zeta}}{\overline{\zeta} - z\omega^{2k}} - \frac{\zeta}{\zeta - \overline{z}\omega^{-2k}}\right]\frac{\mathrm{d}\zeta}{\zeta} \\ &\quad + \frac{1}{2\pi i}\int\limits_{[0,1]\cup[\omega,0]} \gamma(\zeta)\sum_{k=0}^{n-1}\left[\frac{1}{\zeta - z\omega^{2k}} - \frac{1}{\zeta - \overline{z}\omega^{-2k}}\right. \\ &\qquad\qquad \left. - \frac{z}{\omega^{-2k} - z\zeta} + \frac{\overline{z}}{\omega^{2k} - \overline{z}\zeta}\right]\mathrm{d}\zeta \\ &\qquad\qquad - \frac{1}{\pi}\int\limits_{\Omega} f(\zeta)G_1(z,\zeta)\mathrm{d}\xi\mathrm{d}\eta, \qquad z \in \Omega. \end{aligned}$$

Let

$$\widehat{G}_2(z,\zeta) = -\frac{1}{\pi}\int\limits_{\Omega} G_1(z,\widetilde{\zeta})G_1(\widetilde{\zeta},\zeta)\mathrm{d}\widetilde{\xi}\mathrm{d}\widetilde{\eta}, \quad z,\ \zeta \in \Omega, \tag{4.2}$$

with $G_1(z,\zeta)$ defined by (4.1). Then, $\widehat{G}_2(.,\zeta)$ is the solution to the Dirichlet problem

$$\partial_z\partial_{\bar{z}}\widehat{G}_2(z,\zeta) = G_1(z,\zeta) \ \text{ in } \ \Omega, \quad \widehat{G}_2(z,\zeta) = 0 \ \text{ on } \ \partial\Omega. \tag{4.3}$$

Moreover, for any fixed $\zeta \in \Omega$, $\widehat{G}_2(z,\zeta)$ satisfies the properties,

1. $\widehat{G}_2(z,\zeta)$ is biharmonic in $\Omega\backslash\{\zeta\}$,
2. $\widehat{G}_2(z,\zeta) + |\zeta - z|^2\log|\zeta - z|^2$ is biharmonic in $\Omega$,
3. $\widehat{G}_2(z,\zeta) = 0$, $\partial_z\partial_{\bar{z}}\widehat{G}_2(z,\zeta) = 0$ for $z \in \partial\Omega$,
4. $\widehat{G}_2(z,\zeta) = \widehat{G}_2(\zeta,z)$ for $z \neq \zeta$.

Since the boundary $\partial\Omega$ consists of a circular arc and two lines, it is difficult to obtain $\widehat{G}_2(z,\zeta)$ explicitly by direct computation as in the unit disc. Therefore, we prefer to transform $\widehat{G}_2(z,\zeta)$ into a new unknown function. From the property 2 of $\widehat{G}_2(z,\zeta)$, we see that $\widehat{G}_2(z,\zeta)$ can be represented as

$$\widehat{G}_2(z,\zeta) = |\zeta - z|^2G_1(z,\zeta) + h_2(z,\zeta), \quad z,\ \zeta \in \Omega, \tag{4.4}$$

where $h_2(z,\zeta)$ is a biharmonic function in $\Omega$. Thus from (4.3),

$$\partial_z\partial_{\overline{z}}h_2(z,\zeta) = 2\mathrm{Re}[(\zeta - z)\partial_z G_1(z,\zeta)] \ \text{ in } \Omega, \quad h_2(z,\zeta) = 0 \ \text{ on } \ \partial\Omega.$$

Then from Lemma 4.1.1, we obtain

$$h_2(z,\zeta) = \frac{1}{\pi}\int\limits_{\Omega} F(\widetilde{\zeta},\ \zeta)G_1(z,\widetilde{\zeta})\mathrm{d}\widetilde{\xi}\mathrm{d}\widetilde{\eta}, \tag{4.5}$$

where

$$\begin{aligned} F(\widetilde{\zeta},\zeta) \ &= \sum_{k=0}^{n-1}\Bigg[\frac{\zeta - \zeta\omega^{2k}}{\zeta - \widetilde{\zeta}\omega^{2k}} + \frac{\overline{\zeta} - \overline{\zeta}\omega^{-2k}}{\overline{\zeta} - \overline{\overline{\widetilde{\zeta}}}\omega^{-2k}} + \frac{|\zeta|^2 - \omega^{-2k}}{\omega^{-2k} - \widetilde{\zeta}\overline{\zeta}} + \frac{|\zeta|^2 - \omega^{2k}}{\omega^{2k} - \overline{\overline{\widetilde{\zeta}}}\zeta} \\ &\quad + \frac{\zeta\omega^{2k} - \overline{\zeta}}{\overline{\zeta} - \widetilde{\zeta}\omega^{2k}} + \frac{\overline{\zeta}\omega^{-2k} - \zeta}{\zeta - \overline{\overline{\widetilde{\zeta}}}\omega^{-2k}} + \frac{\omega^{2k} - \overline{\zeta}^2}{\omega^{2k} - \overline{\overline{\widetilde{\zeta}}}\overline{\zeta}} + \frac{\omega^{-2k} - \zeta^2}{\omega^{-2k} - \widetilde{\zeta}\zeta}\Bigg]. \end{aligned}$$

Define a new biharmonic function

$$\begin{aligned} H_0(z,\zeta) \ &= \sum_{k=0}^{n-1}\Bigg[\left(z\overline{\zeta} + \overline{z}\zeta - z\overline{\zeta}\omega^{2k} - \overline{z}\zeta\omega^{-2k}\right)\log|\zeta - z\omega^{2k}|^2 \\ &\quad + \left(\frac{z}{\zeta}\omega^{2k} + \frac{\overline{z}}{\overline{\zeta}}\omega^{-2k} - z\overline{\zeta} - \overline{z}\zeta\right)\log|\omega^{-2k} - z\overline{\zeta}|^2 \\ &\quad + \left(z\zeta\omega^{2k} + \overline{z}\overline{\zeta}\omega^{-2k} - z\overline{\zeta} - \overline{z}\zeta\right)\log|\zeta - \overline{z}\omega^{-2k}|^2 \\ &\quad + \left(z\overline{\zeta} + \overline{z}\zeta - \frac{z}{\overline{\zeta}}\omega^{2k} - \frac{\overline{z}}{\zeta}\omega^{-2k}\right)\log|\omega^{-2k} - z\zeta|^2\Bigg]. \end{aligned} \tag{4.6}$$

Then, the next result holds.

**Theorem 4.1.1.** *For* $z,\ \zeta \in \Omega$,

$$h_2(z,\zeta) = -H_0(z,\zeta) + J_0(z,\zeta),$$

*where* $H_0(z,\zeta)$ *is given by* (4.6) *and*

$$\begin{aligned} J_0(z,\zeta) \ &= \frac{1 - |\zeta|^2}{2\pi i}\int\limits_{L}\ \sum_{k=0}^{n-1}\Bigg\{\left[\frac{\widetilde{\zeta}}{\zeta}\omega^{2k} + \frac{\overline{\overline{\widetilde{\zeta}}}}{\overline{\zeta}}\omega^{-2k}\right]\log|\zeta - \widetilde{\zeta}\omega^{2k}|^2 \\ &\quad - \left[\frac{\widetilde{\zeta}}{\overline{\zeta}}\omega^{2k} + \frac{\overline{\overline{\widetilde{\zeta}}}}{\zeta}\omega^{-2k}\right]\log|\zeta - \overline{\overline{\widetilde{\zeta}}}\omega^{-2k}|^2\Bigg\} \\ &\quad \times \sum_{k=0}^{n-1}\left[\frac{\widetilde{\zeta}}{\widetilde{\zeta} - z\omega^{2k}} + \frac{\overline{\overline{\widetilde{\zeta}}}}{\overline{\overline{\widetilde{\zeta}}} - \overline{z}\omega^{-2k}} - \frac{\overline{\overline{\widetilde{\zeta}}}}{\overline{\overline{\widetilde{\zeta}}} - z\omega^{2k}} - \frac{\widetilde{\zeta}}{\widetilde{\zeta} - \overline{z}\omega^{-2k}}\right]\frac{\mathrm{d}\widetilde{\zeta}}{\widetilde{\zeta}}. \end{aligned} \tag{4.7}$$

**Proof:** We can easily verify that

$$\partial_z\partial_{\overline{z}}H_0(z,\zeta)=F(z,\zeta),\quad z,\ \zeta\in\Omega.$$

What is more, for $z\in L$,

$$\begin{aligned} H_0(z,\zeta) &= \sum_{k=0}^{n-1}\Bigg\{\left(\frac{z}{\zeta}\omega^{2k}+\frac{\overline{z}}{\overline{\zeta}}\omega^{-2k}-z\overline{\zeta}\omega^{2k}-\overline{z}\zeta\omega^{-2k}\right)\log|\zeta-z\omega^{2k}|^2 \\ &\quad+\left(z\zeta\omega^{2k}+\overline{z}\overline{\zeta}\omega^{-2k}-\frac{z}{\overline{\zeta}}\omega^{2k}-\frac{\overline{z}}{\zeta}\omega^{-2k}\right)\log|\zeta-\overline{z}\omega^{-2k}|^2\Bigg\}, \end{aligned} \tag{4.8}$$

and when $z\in[0,\omega]$, that is, $z=\rho e^{i\theta},\ 0\le\rho\le 1$,

$$\begin{aligned} H_0(z,\zeta) &= \sum_{k=0}^{n-1}\Bigg[\left(\rho\overline{\zeta}e^{i\theta}+\rho\zeta e^{-i\theta}-\rho\overline{\zeta}\omega^{-(2k-1)}-\rho\zeta\omega^{2k-1}\right)\log|\zeta-\rho\omega^{-(2k-1)}|^2 \\ &\quad+\left(\frac{\rho\omega^{-(2k-1)}}{\zeta}+\frac{\rho\omega^{2k-1}}{\overline{\zeta}}-\rho\overline{\zeta}e^{i\theta}-\rho\zeta e^{-i\theta}\right)\log|\omega^{2k-1}-\rho\overline{\zeta}|^2 \\ &\quad+\left(\rho\zeta\omega^{2k+1}+\rho\overline{\zeta}\omega^{-(2k+1)}-\rho\overline{\zeta}e^{i\theta}-\rho\zeta e^{-i\theta}\right)\log|\zeta-\rho\omega^{-(2k+1)}|^2 \\ &\quad+\left(\rho\overline{\zeta}e^{i\theta}+\rho\zeta e^{-i\theta}-\frac{\rho\omega^{2k+1}}{\overline{\zeta}}-\frac{\rho\omega^{-(2k+1)}}{\zeta}\right)\log|\omega^{-(2k+1)}-\rho\zeta|^2\Bigg] \\ &= 0. \end{aligned}$$

Obviously, we also obtain $H_0(z,\zeta)=0$ for $z\in[0,1]$. Thus, again by Lemma 4.1.1 and (4.8), the proof is completed.

Now we only need to compute the boundary integral in Theorem 4.1.1. In fact, we observe that $J_0(z,\zeta)$ can be converted into an integral on the whole unit circle. That is

$$\begin{aligned} J_0(z,\zeta) &= \frac{1-|\zeta|^2}{2\pi i}\int\limits_{|\widetilde{\zeta}|=1}\sum_{k=0}^{n-1}\left[\frac{\widetilde{\zeta}}{\widetilde{\zeta}-z\omega^{2k}}+\frac{\overline{\widetilde{\zeta}}}{\overline{\widetilde{\zeta}}-\overline{z}\omega^{-2k}}\right] \\ &\quad\times\left[\left(\frac{\widetilde{\zeta}}{\zeta}+\frac{\overline{\widetilde{\zeta}}}{\overline{\zeta}}\right)\log|\zeta-\widetilde{\zeta}|^2-\left(\frac{\widetilde{\zeta}}{\overline{\zeta}}+\frac{\overline{\widetilde{\zeta}}}{\zeta}\right)\log|\zeta-\overline{\widetilde{\zeta}}|^2\right]\frac{d\widetilde{\zeta}}{\widetilde{\zeta}} \\ &= 2\mathrm{Re}\Bigg\{\frac{1-|\zeta|^2}{2\pi i}\int\limits_{|\widetilde{\zeta}|=1}\sum_{k=0}^{n-1}\left(\frac{\widetilde{\zeta}}{\widetilde{\zeta}-z\omega^{2k}}+\frac{\overline{\widetilde{\zeta}}}{\overline{\widetilde{\zeta}}-\overline{z}\omega^{-2k}}\right) \\ &\quad\times\left[\frac{1}{\zeta}\log|\zeta-\widetilde{\zeta}|^2-\frac{1}{\overline{\zeta}}\log|\zeta-\overline{\widetilde{\zeta}}|^2\right]d\widetilde{\zeta}\Bigg\}. \end{aligned} \tag{4.9}$$

Since

$$\frac{1}{2\pi i}\int\limits_{|\widetilde{\zeta}|=1}\frac{\widetilde{\zeta}}{\widetilde{\zeta}-z\omega^{2k}}\log|\zeta-\widetilde{\zeta}|^2 \mathrm{d}\widetilde{\zeta}=z\omega^{2k}\log(1-z\overline{\zeta}\omega^{2k})-\zeta, \tag{4.10}$$

$$\frac{1}{2\pi i}\int\limits_{|\widetilde{\zeta}|=1}\frac{\widetilde{\zeta}}{\widetilde{\zeta}-z\omega^{2k}}\log|\zeta-\overline{\widetilde{\zeta}}|^2 \mathrm{d}\widetilde{\zeta}=z\omega^{2k}\log(1-z\zeta\omega^{2k})-\overline{\zeta}, \tag{4.11}$$

$$\frac{1}{2\pi i}\int\limits_{|\widetilde{\zeta}|=1}\frac{\overline{\widetilde{\zeta}}\log|\zeta-\widetilde{\zeta}|^2}{\overline{\widetilde{\zeta}}-\overline{z}\omega^{-2k}}\mathrm{d}\widetilde{\zeta}=\frac{1}{\overline{z}}\omega^{2k}\log(1-\overline{z}\zeta\omega^{-2k}), \tag{4.12}$$

$$\frac{1}{2\pi i}\int\limits_{|\widetilde{\zeta}|=1}\frac{\overline{\widetilde{\zeta}}\log|\zeta-\overline{\widetilde{\zeta}}|^2}{\overline{\widetilde{\zeta}}-\overline{z}\omega^{-2k}}\mathrm{d}\widetilde{\zeta}=\frac{1}{\overline{z}}\omega^{2k}\log(1-\overline{z}\overline{\zeta}\omega^{-2k}), \tag{4.13}$$

then from (4.9)-(4.13), we have

$$\begin{aligned}J_0(z,\zeta)=(1-|\zeta|^2)\sum_{k=0}^{n-1}\Bigg\{&\left[\frac{z}{\zeta}\omega^{2k}+\frac{\omega^{-2k}}{z\overline{\zeta}}\right]\log(1-z\overline{\zeta}\omega^{2k})\\ &+\left[\frac{\omega^{2k}}{\overline{z}\zeta}+\frac{\overline{z}}{\overline{\zeta}}\omega^{-2k}\right]\log(1-\overline{z}\zeta\omega^{-2k})-\left[\frac{z\omega^{2k}}{\overline{\zeta}}+\frac{\omega^{-2k}}{z\zeta}\right]\log(1-z\zeta\omega^{2k})\\ &-\left[\frac{\omega^{2k}}{\overline{z}\overline{\zeta}}+\frac{\overline{z}\omega^{-2k}}{\zeta}\right]\log(1-\overline{z}\overline{\zeta}\omega^{-2k})\Bigg\}.\end{aligned} \tag{4.14}$$

By (4.6), (4.14) and Theorem 4.1.1,

$$\begin{aligned}h_2(z,\zeta)=\sum_{k=0}^{n-1}\Bigg\{&\left(z\overline{\zeta}\omega^{2k}+\overline{z}\zeta\omega^{-2k}-z\overline{\zeta}-\overline{z}\zeta\right)\log|\zeta-z\omega^{2k}|^2\\ &+\left(z\overline{\zeta}+\overline{z}\zeta-\overline{z}\overline{\zeta}\omega^{-2k}-z\zeta\omega^{2k}\right)\log|\zeta-\overline{z}\omega^{-2k}|^2\\ &+\left(z\overline{\zeta}+\overline{z}\zeta-\frac{\overline{z}}{\overline{\zeta}}\omega^{-2k}+\frac{\omega^{-2k}}{z\overline{\zeta}}-z\overline{\zeta}\omega^{2k}-\frac{\zeta}{z}\omega^{-2k}\right)\log(1-z\overline{\zeta}\omega^{2k})\\ &+\left(z\overline{\zeta}+\overline{z}\zeta-\frac{\overline{\zeta}}{\overline{z}}\omega^{2k}+\frac{\omega^{2k}}{\overline{z}\zeta}-\overline{z}\zeta\omega^{-2k}-\frac{z}{\zeta}\omega^{2k}\right)\log(1-\overline{z}\zeta\omega^{-2k})\\ &-\left(z\overline{\zeta}+\overline{z}\zeta-\frac{\overline{z}}{\zeta}\omega^{-2k}+\frac{\omega^{-2k}}{z\zeta}-z\zeta\omega^{2k}-\frac{\overline{\zeta}}{z}\omega^{-2k}\right)\log(1-z\zeta\omega^{2k})\\ &-\left(z\overline{\zeta}+\overline{z}\zeta-\frac{z}{\overline{\zeta}}\omega^{2k}+\frac{\omega^{2k}}{\overline{z}\overline{\zeta}}-\overline{z}\overline{\zeta}\omega^{-2k}-\frac{\zeta}{\overline{z}}\omega^{2k}\right)\log(1-\overline{z}\overline{\zeta}\omega^{-2k})\Bigg\}.\end{aligned} \tag{4.15}$$

Hence, by (4.4) and simple computation,

$$\widehat{G}_2(z,\zeta)=\sum_{k=0}^{n-1}\Bigg\{\left|\zeta-\overline{z}\omega^{-2k}\right|^2\log\left|\frac{\zeta-\overline{z}\omega^{-2k}}{1-z\zeta\omega^{2k}}\right|^2-\left|\zeta-z\omega^{2k}\right|^2\log\left|\frac{\zeta-z\omega^{2k}}{1-\overline{z}\zeta\omega^{-2k}}\right|^2$$
$$+(|z|^2-1)(|\zeta|^2-1)\Bigg[\frac{\log(1-z\overline{\zeta}\omega^{2k})}{z\overline{\zeta}\omega^{2k}}+\frac{\log(1-\overline{z}\zeta\omega^{-2k})}{\overline{z}\zeta\omega^{-2k}}$$
$$-\frac{\log(1-z\zeta\omega^{2k})}{z\zeta\omega^{2k}}-\frac{\log(1-\overline{z}\overline{\zeta}\omega^{-2k})}{\overline{z}\overline{\zeta}\omega^{-2k}}\Bigg]\Bigg\}. \tag{4.16}$$

Finally, we can verify that expression (4.16) is exactly the solution to the problem

$$\partial_z\partial_{\overline{z}}\widehat{G}_2(z,\zeta)=G_1(z,\zeta)\ \text{ in }\ \Omega,\quad \widehat{G}_2(z,\zeta)=0\ \text{ on }\ \partial\Omega.$$

**Remark 4.1.2.** In particular, when $n=1$, the biharmonic Green function for $\mathbb{D}^+$ (half unit disc) is

$$\widehat{G}_2(z,\zeta)\ =|\zeta-\overline{z}|^2\log\left|\frac{\zeta-\overline{z}}{1-z\zeta}\right|^2-|\zeta-z|^2\log\left|\frac{\zeta-z}{1-\overline{z}\zeta}\right|^2$$
$$+(|\zeta|^2-1)(|z|^2-1)\Bigg[\frac{\log(1-z\overline{\zeta})}{z\overline{\zeta}}+\frac{\log(1-\overline{z}\zeta)}{\overline{z}\zeta}$$
$$-\frac{\log(1-z\zeta)}{z\zeta}-\frac{\log(1-\overline{z}\overline{\zeta})}{\overline{z}\overline{\zeta}}\Bigg].$$

## 4.2 Biharmonic Neumann Function

Similarly to the biharmonic Green function, we construct a biharmonic Neumann function based on the convolution of the harmonic Neumann function with itself. By the conclusion in Section 3.2 and Remark 4.1.1, we know the harmonic Neumann function for the domain $\Omega$ can be rewritten as

$$N_1(z,\zeta)=-\sum_{k=0}^{n-1}\log\left|\left(\zeta-z\omega^{2k}\right)\left(\overline{\zeta}-z\omega^{2k}\right)\left(\omega^{-2k}-z\zeta\right)\left(\omega^{-2k}-z\overline{\zeta}\right)\right|^2, \tag{4.17}$$
$$z,\ \zeta\in\Omega.$$

Especially, the boundary behavior and normalization condition are, respectively,

$$\partial_{\nu_z}N_1(z,\zeta)=\begin{cases}-4n, & z\in L\backslash\{1,\omega\},\\ 0, & z\in(0,1)\cup(\omega,0),\end{cases}\qquad \text{for}\quad \zeta\in\Omega,$$

and

$$\int\limits_L N_1(z,\zeta)\frac{\mathrm{d}z}{z}=0.$$

In addition, the following result for the Neumann problem is true.

**Lemma 4.2.1.** *The Neumann problem for the domain* $\Omega$

$$\partial_z\partial_{\overline{z}}w=f \quad in \quad \Omega, \qquad \partial_{\nu_z}w=\gamma \quad on \quad \partial\Omega, \qquad \frac{n}{\pi}\int_0^{\pi/n} w(e^{i\theta})\mathrm{d}\theta=c_0$$

*for* $f\in L_p(\Omega;\mathbb{C}),\ p>2,\ \gamma\in C(\partial\Omega;\mathbb{C}),\ c_0\in\mathbb{C}$ *is solvable if and only if*

$$\frac{1}{2\pi}\int\limits_{\partial\Omega}\gamma(\zeta)\mathrm{d}s_\zeta=\frac{2}{\pi}\int\limits_{\Omega}f(\zeta)\mathrm{d}\xi\mathrm{d}\eta. \tag{4.18}$$

*Then the solution is uniquely expressed by*

$$\begin{aligned} w(z) &= c_0-\frac{1}{2\pi i}\int\limits_L \gamma(\zeta)\sum_{k=0}^{n-1}\log|\big(\zeta-z\omega^{2k}\big)\big(\overline{\zeta}-z\omega^{2k}\big)|^2\frac{\mathrm{d}\zeta}{\zeta}\\ &\quad -\frac{1}{2\pi}\int_0^1\gamma(\zeta)\sum_{k=0}^{n-1}\log|\big(\zeta-z\omega^{2k}\big)\big(1-z\zeta\omega^{2k}\big)|^2\mathrm{d}\zeta\\ &\quad +\frac{1}{2\pi}\int\limits_{[\omega,0]}e^{-i\theta}\gamma(\zeta)\sum_{k=0}^{n-1}\log|\big(\zeta-z\omega^{2k}\big)\big(1-z\zeta\omega^{2k}\big)|^2\mathrm{d}\zeta\\ &\quad -\frac{1}{\pi}\int\limits_{\Omega}f(\zeta)N_1(z,\zeta)\mathrm{d}\xi\mathrm{d}\eta,\quad z\in\Omega, \end{aligned} \tag{4.19}$$

*where* $N_1(z,\zeta)$ *is defined by* (4.17).

Similarly,

$$N_2(z,\zeta)=-\frac{1}{\pi}\int\limits_{\Omega}N_1(z,\widetilde{\zeta})N_1(\widetilde{\zeta},\zeta)\mathrm{d}\widetilde{\xi}\mathrm{d}\widetilde{\eta},\quad z,\ \zeta\in\Omega$$

is introduced. Applying Lemma 4.2.1 to $w(z)=|z|^2-1$ , we easily obtain

$$-\frac{1}{\pi}\int\limits_{\Omega}N_1(z,\zeta)\mathrm{d}\xi\mathrm{d}\eta=|z|^2-1,\quad z\in\Omega.$$

Then, from the above properties of $N_1(z,\zeta)$, $N_2(.,\zeta)$ satisfies

1. $N_2(z,\zeta)$ is biharmonic in $\Omega\backslash\{\zeta\}$,

2. $N_2(z,\zeta)+|\zeta-z|^2\log|\zeta-z|^2$ is biharmonic in $\Omega$,

3. $\partial_z\partial_{\overline{z}}N_2(z,\zeta)=N_1(z,\zeta),\ z,\zeta\in\Omega$,

4. $\partial_{\nu_z}N_2(z,\zeta)=\begin{cases}\dfrac{4n}{\pi}\displaystyle\int_\Omega N_1(\widetilde{\zeta},\zeta)\mathrm{d}\widetilde{\xi}\mathrm{d}\widetilde{\eta}=-4n(|\zeta|^2-1), & z\in L\backslash\{1,\ \omega\},\\ 0, & z\in(0,1)\cup(\omega,0),\end{cases}$

5. Normalization condition $\displaystyle\int_L N_2(z,\zeta)\frac{\mathrm{d}z}{z}=0.$

Thus, $N_2(z,\zeta)$ can be rewritten as

$$N_2(z,\zeta)=|\zeta-z|^2N_1(z,\zeta)+\widehat{h}_2(z,\zeta),\quad z,\zeta\in\Omega, \tag{4.20}$$

with $\widehat{h}_2(z,\zeta)$ being a biharmonic function in $\Omega$.

From the properties 3-4 of $N_2(z,\zeta)$ and (4.20), we obtain,

$$\begin{aligned}\partial_z\partial_{\overline{z}}\widehat{h}_2(z,\zeta)&=(\zeta-z)\partial_zN_1(z,\zeta)+(\overline{\zeta}-\overline{z})\partial_{\overline{z}}N_1(z,\zeta)\\ &=8n+2\sum_{k=0}^{n-1}\mathrm{Re}\left\{\frac{|\zeta|^2-\omega^{-2k}}{\omega^{-2k}-z\overline{\zeta}}+\frac{\zeta\omega^{2k}-\overline{\zeta}}{\overline{\zeta}-z\omega^{2k}}+\frac{\zeta^2-\omega^{-2k}}{\omega^{-2k}-z\zeta}+\frac{\zeta\omega^{2k}-\zeta}{\zeta-z\omega^{2k}}\right\},\\ &\qquad z,\zeta\in\Omega,\end{aligned}\tag{4.21}$$

and

$$\partial_{\nu_z}\widehat{h}_2(z,\zeta)=\begin{cases}(z\partial_z+\overline{z}\partial_{\overline{z}})\widehat{h}_2(z,\zeta)=(z\overline{\zeta}+\overline{z}\zeta-2)[N_1(z,\zeta)-4n],\\ \qquad\qquad z\in L\backslash\{1,\omega\},\\ -i(\partial_z-\partial_{\overline{z}})\widehat{h}_2(z,\zeta)=-i(\overline{\zeta}-\zeta)N_1(z,\zeta),\quad z\in(0,1),\\ i(e^{i\theta}\partial_z-e^{-i\theta}\partial_{\overline{z}})\widehat{h}_2(z,\zeta)=i(\overline{\zeta}e^{i\theta}-\zeta e^{-i\theta})N_1(z,\zeta),\quad z\in(\omega,0).\end{cases}\tag{4.22}$$

Let $f_0(z,\zeta)$ be the expression on the right-hand side in (4.21) and suppose

$$\begin{aligned}F_0(z,\zeta)=8n|z|^2+(z\overline{\zeta}+\overline{z}\zeta)N_1(z,\zeta)+\sum_{k=0}^{n-1}\Bigg\{&\left(z\overline{\zeta}\omega^{2k}+\overline{z}\zeta\omega^{-2k}\right)\log|\zeta-z\omega^{2k}|^2\\ +\left(\frac{z}{\zeta}\omega^{2k}+\frac{\overline{z}}{\overline{\zeta}}\omega^{-2k}\right)\log|\omega^{-2k}-z\overline{\zeta}|^2&+\left(\frac{z}{\overline{\zeta}}\omega^{2k}+\frac{\overline{z}}{\zeta}\omega^{-2k}\right)\log|\omega^{-2k}-z\zeta|^2\\ +\left(z\zeta\omega^{2k}+\overline{z}\overline{\zeta}\omega^{-2k}\right)\log|\zeta-\overline{z}\omega^{-2k}|^2\Bigg\},&\quad z,\zeta\in\Omega.\end{aligned}\tag{4.23}$$

Then we can verify that

$$\partial_z \partial_{\overline{z}} F_0(z,\zeta) = f_0(z,\zeta) \quad \text{for} \quad z,\ \zeta \in \Omega, \tag{4.24}$$

and for $z \in L\backslash\{1,\omega\}$,

$$\begin{aligned}
\partial_{\nu_z} F_0(z,\zeta) &= (z\partial_z + \overline{z}\partial_{\overline{z}})F_0(z,\zeta) \\
&= -4n(z\overline{\zeta} + \overline{z}\zeta) + 16n + 2\sum_{k=0}^{n-1} \mathrm{Re}\Bigg\{ 2(z\zeta\omega^{2k} + \overline{z}\zeta\omega^{-2k}) \\
&\quad + \left(\frac{z\omega^{2k}}{\zeta} + z\overline{\zeta}\omega^{2k} - 2z\overline{\zeta}\right) \log|\zeta - z\omega^{2k}|^2 \\
&\quad + \left(\frac{\overline{z}\omega^{-2k}}{\zeta} + \overline{z}\overline{\zeta}\omega^{-2k} - 2\overline{z}\zeta\right) \log|\overline{\zeta} - z\omega^{2k}|^2 \\
&\quad + \frac{\zeta\left(z\omega^{2k}/\zeta + \overline{z}\omega^{-2k}/\overline{\zeta} - z\overline{\zeta}\omega^{2k} - \overline{z}\zeta\omega^{-2k}\right)}{\zeta - z\omega^{2k}} \\
&\quad + \frac{\zeta\left(z\omega^{2k}/\overline{\zeta} + \overline{z}\omega^{-2k}/\zeta - \overline{z}\overline{\zeta}\omega^{-2k} - z\zeta\omega^{2k}\right)}{\zeta - \overline{z}\omega^{-2k}} \Bigg\}.
\end{aligned} \tag{4.25}$$

Also for $z \in (0,1)$,

$$\begin{aligned}
\partial_{\nu_z} F_0(z,\zeta) &= -i(\partial_z - \partial_{\overline{z}})F_0(z,\zeta) \\
&= 2i(\overline{\zeta} - \zeta)\sum_{k=0}^{n-1} \log\left|(\omega^{-2k} - z\overline{\zeta})(\zeta - z\omega^{2k})\right|^2,
\end{aligned} \tag{4.26}$$

and for $z \in (\omega, 0)$, as $ze^{-i\theta} = \overline{z}e^{i\theta}$, then

$$\begin{aligned}
\partial_{\nu_z} F_0(z,\zeta) &= i(e^{i\theta}\partial_z - e^{-i\theta}\partial_{\overline{z}})F_0(z,\zeta) \\
&= -2i(\overline{\zeta}e^{i\theta} - \zeta e^{-i\theta})\sum_{k=0}^{n-1} \log\left|(\omega^{-2k} - z\overline{\zeta})(\overline{\zeta} - z\omega^{2k})\right|^2.
\end{aligned} \tag{4.27}$$

By (4.22) and (4.25), we obtain for $z \in L\backslash\{1,\omega\}$,

$$\partial_{\nu_z}\widehat{h}_2(z,\zeta) - \partial_{\nu_z}F_0(z,\zeta) = -8n - 2\mathrm{Re}\left\{\Delta_1(z,\zeta) + \Delta_2(z,\zeta) + \Delta_3(z,\zeta)\right\}, \tag{4.28}$$

where

$$\begin{aligned}
\Delta_1(z,\zeta) &= \sum_{k=0}^{n-1}\Bigg[ \left(\frac{z\omega^{2k}}{\zeta} + z\overline{\zeta}\omega^{2k} - 2\right)\log|\zeta - z\omega^{2k}|^2 \\
&\quad + \left(\frac{\overline{z}\omega^{-2k}}{\zeta} + \overline{z}\overline{\zeta}\omega^{-2k} - 2\right)\log|\overline{\zeta} - z\omega^{2k}|^2 \Bigg],
\end{aligned} \tag{4.29}$$

$$\Delta_2(z,\zeta) = \sum_{k=0}^{n-1}\left[\frac{\zeta\left(z\omega^{2k}/\zeta+\overline{z}\omega^{-2k}/\overline{\zeta}-z\overline{\zeta}\omega^{2k}-\overline{z}\zeta\omega^{-2k}\right)}{\zeta-z\omega^{2k}}\right. \\ \left.+\frac{\zeta\left(z\omega^{2k}/\overline{\zeta}+\overline{z}\omega^{-2k}/\zeta-\overline{z}\overline{\zeta}\omega^{-2k}-z\zeta\omega^{2k}\right)}{\zeta-\overline{z}\omega^{-2k}}\right], \tag{4.30}$$

$$\Delta_3(z,\zeta) = 2\sum_{k=0}^{n-1}(z\zeta+\overline{z}\zeta)\omega^{2k}. \tag{4.31}$$

Hence, the following result is valid.

**Theorem 4.2.1.** *For $z,\zeta\in\Omega$,*

$$\widehat{h}_2(z,\zeta) = 4n(|\zeta|^2-1)+F_0(z,\zeta) \\ -\mathrm{Re}\left\{\frac{1}{2\pi i}\int_L\left[\Delta_1(\widetilde{\zeta},\zeta)+\Delta_2(\widetilde{\zeta},\zeta)+\Delta_3(\widetilde{\zeta},\zeta)\right]N_1(z,\widetilde{\zeta})\frac{\mathrm{d}\widetilde{\zeta}}{\widetilde{\zeta}}\right\}.$$

*where $F_0(z,\zeta)$, $\Delta_1,\Delta_2$, $\Delta_3$ are determined by* (4.23), (4.29), (4.30), (4.31) *respectively.*

**Proof:** Let $\widehat{F}(z,\zeta)=\widehat{h}_2(z,\zeta)-F_0(z,\zeta)$, then from (4.22), (4.24), (4.26) and (4.27), we obtain

$$\partial_z\partial_{\overline{z}}\widehat{F}(z,\zeta)=0 \ \text{ in } \ \Omega, \quad \partial_{\nu_z}\widehat{F}(z,\zeta)=0 \ \text{ for } \ z\in(0,1)\cup(\omega,0). \tag{4.32}$$

Since

$$\frac{1}{2\pi i}\int_L[\Delta_1(z,\zeta)+\Delta_2(z,\zeta)+\Delta_3(z,\zeta)]\frac{\mathrm{d}z}{z} \\ = \frac{1}{2\pi i}\int_{|z|=1}\left\{\left[\frac{z}{\zeta}+z\overline{\zeta}-2\right]\log|\zeta-z|^2+\frac{\zeta(z/\zeta+\overline{z}/\overline{\zeta}-z\overline{\zeta}-\overline{z}\zeta)}{\zeta-z}+2z\zeta\right\}\frac{\mathrm{d}z}{z}=-2,$$

and

$$\frac{1}{2\pi i}\int_L(-8n)\frac{\mathrm{d}z}{z}=-4,$$

then by (4.28) and (4.32),

$$\frac{1}{2\pi i}\int_L\partial_{\nu_z}\widehat{F}(z,\zeta)\frac{\mathrm{d}z}{z}=0 \implies \frac{1}{2\pi}\int_{\partial\Omega}\partial_{\nu_z}\widehat{F}(z,\zeta)\mathrm{d}s_z=0,$$

which implies that the solvability condition (4.18) for $\widehat{F}(z,\zeta)$ is true. Moreover, by (4.20), (4.23) and the normalization of $N_1(z,\zeta),\ N_2(z,\zeta)$,

$$\begin{aligned} c_0 &= \frac{n}{\pi i}\int\limits_L \widehat{F}(z,\zeta)\frac{\mathrm{d}z}{z} \\ &= -8n - 2\mathrm{Re}\left\{\frac{(1+|\zeta|^2)n}{\zeta\pi i}\int\limits_{|z|=1}\log|\zeta-z|^2\mathrm{d}z\right\} = 4n(|\zeta|^2-1). \end{aligned}$$

Then, from Lemma 4.2.1,

$$\widehat{F}(z,\zeta) = 4n(|\zeta|^2-1) + \frac{1}{4\pi i}\int\limits_L \left[\partial_{\nu_{\widetilde{\zeta}}}\widehat{h}_2(\widetilde{\zeta},\zeta) - \partial_{\nu_{\widetilde{\zeta}}}F_0(\widetilde{\zeta},\zeta)\right] N_1(z,\widetilde{\zeta})\frac{\mathrm{d}\widetilde{\zeta}}{\widetilde{\zeta}}.$$

Thus, the desired result follows from (4.28) and the normalization of $N_1(z,\zeta)$.

Let $B_0 = \sum_{k=0}^{n-1}\omega^{2k}$, then we have

$$B_0 = \begin{cases} 1, & n=1, \\ 0, & n>1. \end{cases} \tag{4.33}$$

Suppose

$$f(\widetilde{\zeta}) = \left(\frac{1}{\overline{\zeta}} + \zeta - 2\widetilde{\zeta}\right)\log\left[(1-\overline{z}\widetilde{\zeta}\omega^{2k})(1-z\widetilde{\zeta}\omega^{2k})\right],$$

and

$$g(\widetilde{\zeta}) = \left(\frac{\widetilde{\zeta}}{\zeta} + \widetilde{\zeta}\overline{\zeta} - 2\right)\log\left[(1-\overline{z}\widetilde{\zeta}\omega^{2k})(1-z\widetilde{\zeta}\omega^{2k})\right].$$

Then, the following lemmas are valid.

**Lemma 4.2.2.** *For* $l = 0, 1, \cdots,$

$$\begin{aligned} f^{(l+2)}(\widetilde{\zeta}) &= 2(l+2)l!\left[\frac{(\overline{z}\omega^{2k})^{l+1}}{(1-\overline{z}\widetilde{\zeta}\omega^{2k})^{l+1}} + \frac{(z\omega^{2k})^{l+1}}{(1-z\widetilde{\zeta}\omega^{2k})^{l+1}}\right] \\ &\quad -\left(\frac{1}{\overline{\zeta}} + \zeta - 2\widetilde{\zeta}\right)(l+1)!\left[\frac{(\overline{z}\omega^{2k})^{l+2}}{(1-\overline{z}\widetilde{\zeta}\omega^{2k})^{l+2}} + \frac{(z\omega^{2k})^{l+2}}{(1-z\widetilde{\zeta}\omega^{2k})^{l+2}}\right]. \end{aligned} \tag{4.34}$$

**Proof:** When $l = 0$, we obviously obtain

$$
\begin{aligned}
f^{(2)}(\widetilde{\zeta}) &= 4\left[\frac{\overline{z}\omega^{2k}}{1-\overline{z}\widetilde{\zeta}\omega^{2k}} + \frac{z\omega^{2k}}{1-z\widetilde{\zeta}\omega^{2k}}\right] \\
&\quad -\Big(\frac{1}{\overline{\overline{\zeta}}} + \zeta - 2\widetilde{\zeta}\Big)\left[\frac{(\overline{z}\omega^{2k})^2}{(1-\overline{z}\widetilde{\zeta}\omega^{2k})^2} + \frac{(z\omega^{2k})^2}{(1-z\widetilde{\zeta}\omega^{2k})^2}\right].
\end{aligned}
$$

Suppose that when $l = m$, (4.34) is true, then we have for $l = m+1$,

$$
\begin{aligned}
& f^{(m+3)}(\widetilde{\zeta}) = \partial_{\widetilde{\zeta}}[f^{(m+2)}(\widetilde{\zeta})] \\
&= 2[(m+2)! + (m+1)!]\left[\frac{(\overline{z}\omega^{2k})^{m+2}}{(1-\overline{z}\zeta\omega^{2k})^{m+2}} + \frac{(z\omega^{2k})^{m+2}}{(1-z\zeta\omega^{2k})^{m+2}}\right] \\
&\quad -\Big(\frac{1}{\overline{\overline{\zeta}}} + \zeta - 2\widetilde{\zeta}\Big)(m+2)!\left[\frac{(\overline{z}\omega^{2k})^{m+3}}{(1-\overline{z}\widetilde{\zeta}\omega^{2k})^{m+3}} + \frac{(z\omega^{2k})^{m+3}}{(1-z\widetilde{\zeta}\omega^{2k})^{m+3}}\right],
\end{aligned}
$$

which implies that (4.34) is also true for $l = m+1$. This completes the proof.

In exactly the same way, we have the following result.

**Lemma 4.2.3.** *For* $l = 1, 2, \cdots$,

$$
g^{(l)}(\widetilde{\zeta}) = \begin{cases}
\left\{\left(\frac{1}{\zeta} + \overline{\zeta}\right)\log\left[(1-\overline{z}\widetilde{\zeta}\omega^{2k})(1-z\widetilde{\zeta}\omega^{2k})\right]\right. \\
\left.-\left(\frac{\widetilde{\zeta}}{\zeta} + \widetilde{\zeta}\overline{\zeta} - 2\right)\left[\frac{\overline{z}\omega^{2k}}{1-\overline{z}\widetilde{\zeta}\omega^{2k}} + \frac{z\omega^{2k}}{1-z\widetilde{\zeta}\omega^{2k}}\right]\right\}, & l = 1, \\
-\left\{l(l-2)!\left(\frac{1}{\zeta} + \overline{\zeta}\right)\left[\frac{(\overline{z}\omega^{2k})^{l-1}}{(1-\overline{z}\widetilde{\zeta}\omega^{2k})^{l-1}} + \frac{(z\omega^{2k})^{l-1}}{(1-z\widetilde{\zeta}\omega^{2k})^{l-1}}\right]\right. \\
\left.+(l-1)!\left(\frac{\widetilde{\zeta}}{\zeta} + \widetilde{\zeta}\overline{\zeta} - 2\right)\left[\frac{(\overline{z}\omega^{2k})^{l}}{(1-\overline{z}\widetilde{\zeta}\omega^{2k})^{l}} + \frac{(z\omega^{2k})^{l}}{(1-z\widetilde{\zeta}\omega^{2k})^{l}}\right]\right\}, \\
& l = 2, 3, \cdots.
\end{cases}
\tag{4.35}
$$

Next, several integrals are studied.

**Lemma 4.2.4.** *For* $z, \zeta \in \Omega$,

$$
\begin{aligned}
& \frac{1}{2\pi i}\int_L \Delta_2(\widetilde{\zeta},\zeta)N_1(z,\widetilde{\zeta})\frac{d\widetilde{\zeta}}{\widetilde{\zeta}} \\
&= \frac{2(1-|\zeta|^2)}{\overline{\zeta}}(\overline{z}+z)B_0 + \frac{2(1-|\zeta|^4)}{|\zeta|^2}\sum_{k=0}^{n-1}\log\left[(1-\overline{z}\zeta\omega^{-2k})(1-z\zeta\omega^{2k})\right],
\end{aligned}
$$

*where* $\Delta_2$*,* $B_0$ *are defined by* (4.30) *and* (4.33) *respectively.*

**Proof:** Since

$$-\frac{1}{2\pi i}\int\limits_{|\widetilde{\zeta}|=1}\frac{\log\left|(\widetilde{\zeta}-z\omega^{2k})(\overline{\widetilde{\zeta}}-z\omega^{2k})\right|^2}{\zeta-\widetilde{\zeta}}\mathrm{d}\widetilde{\zeta}=\log\left[(1-\overline{z}\zeta\omega^{-2k})(1-z\zeta\omega^{2k})\right],$$

and

$$-\frac{\zeta}{2\pi i}\frac{1-|\zeta|^2}{\overline{\zeta}}\int\limits_{|\widetilde{\zeta}|=1}\frac{\log\left|(\widetilde{\zeta}-z\omega^{-2k})(\overline{\widetilde{\zeta}}-z\omega^{2k})\right|^2}{\zeta-\widetilde{\zeta}}\frac{\mathrm{d}\widetilde{\zeta}}{\widetilde{\zeta}^2}$$
$$=\left(\frac{1}{|\zeta|^2}-1\right)\log(1-\overline{z}\zeta\omega^{2k})(1-z\zeta\omega^{2k})+\frac{1-|\zeta|^2}{\overline{\zeta}}(\overline{z}+z)\omega^{2k},$$

thus, the lemma follows from

$$-\frac{1}{2\pi i}\int\limits_L\Delta_2(\widetilde{\zeta},\zeta)\sum_{k=0}^{n-1}\left[\log\left|\widetilde{\zeta}-z\omega^{2k}\right|^2+\log\left|\overline{\widetilde{\zeta}}-z\omega^{2k}\right|^2\right]\frac{\mathrm{d}\widetilde{\zeta}}{\widetilde{\zeta}}$$
$$=-\sum_{k=0}^{n-1}\left[\frac{1-|\zeta|^2}{2\pi i}\int\limits_{|\widetilde{\zeta}|=1}\zeta\left(\frac{\overline{\widetilde{\zeta}}}{\overline{\zeta}}+\frac{\widetilde{\zeta}}{\zeta}\right)\log\left|(\widetilde{\zeta}-z\omega^{2k})(\overline{\widetilde{\zeta}}-z\omega^{2k})\right|^2\frac{\mathrm{d}\widetilde{\zeta}}{\widetilde{\zeta}(\zeta-\widetilde{\zeta})}\right]$$
$$=\frac{1-|\zeta|^4}{|\zeta|^2}\sum_{k=0}^{n-1}\log\left[(1-\overline{z}\zeta\omega^{-2k})(1-z\zeta\omega^{2k})\right]+\frac{1-|\zeta|^2}{\overline{\zeta}}(\overline{z}+z)B_0.$$

**Lemma 4.2.5.** *For* $z,\zeta\in\Omega$*,*

$$\frac{1}{2\pi i}\int\limits_L\Delta_3(\widetilde{\zeta},\zeta)N_1(z,\widetilde{\zeta})\frac{\mathrm{d}\widetilde{\zeta}}{\widetilde{\zeta}}=4\zeta(\overline{z}+z)B_0,$$

*where* $\Delta_3$*,* $B_0$ *are defined by* (4.31) *and* (4.33) *respectively.*

**Proof:** The result derives from

$$-\frac{1}{2\pi i}\int\limits_L\Delta_3(\widetilde{\zeta},\zeta)\sum_{k=0}^{n-1}2\left[\log\left|\widetilde{\zeta}-z\omega^{2k}\right|^2+\log\left|\overline{\widetilde{\zeta}}-z\omega^{2k}\right|^2\right]\frac{\mathrm{d}\widetilde{\zeta}}{\widetilde{\zeta}}$$
$$=-4\zeta\sum_{k=0}^{n-1}\frac{1}{2\pi i}\int\limits_{|\widetilde{\zeta}|=1}\log\left|(1-z\widetilde{\zeta}\omega^{2k})(1-z\overline{\widetilde{\zeta}}\omega^{2k})\right|^2\mathrm{d}\widetilde{\zeta}$$
$$=-4\zeta\sum_{k=0}^{n-1}\overline{\left[\partial_{\widetilde{\zeta}}\log(1-z\widetilde{\zeta}\omega^{2k})(1-\overline{z}\widetilde{\zeta}\omega^{-2k})\right]_{\widetilde{\zeta}=0}}=4\zeta(\overline{z}+z)B_0.$$

**Lemma 4.2.6.** *For $z, \zeta \in \Omega$,*

$$\frac{1}{2\pi i}\int\limits_L \Delta_1(\widetilde{\zeta},\zeta)N_1(z,\widetilde{\zeta})\frac{\mathrm{d}\widetilde{\zeta}}{\widetilde{\zeta}} = -4n\left(1+|\zeta|^2\right) - \frac{2(1+|\zeta|^2)}{\zeta}(z+\overline{z})B_0$$
$$+\sum_{k=0}^{n-1}\sum_{l=0}^{\infty}\frac{4}{(l+1)^2}\left[(\overline{z}\zeta\omega^{-2k})^{l+1}+(z\zeta\omega^{2k})^{l+1}+(\overline{z}\overline{\zeta}\omega^{-2k})^{l+1}+(z\overline{\zeta}\omega^{2k})^{l+1}\right]$$
$$-2(1+|\zeta|^2)\sum_{k=0}^{n-1}\Bigg\{\frac{(1-\overline{z}\zeta\omega^{-2k})\log(1-\overline{z}\zeta\omega^{-2k})}{\overline{z}\zeta\omega^{-2k}}+\frac{(1-z\zeta\omega^{2k})\log(1-z\zeta\omega^{2k})}{z\zeta\omega^{2k}}$$
$$+\frac{1}{|\zeta|^2}\Big[(1-\overline{z}\overline{\zeta}\omega^{-2k})\log(1-\overline{z}\overline{\zeta}\omega^{-2k})+(1-z\overline{\zeta}\omega^{2k})\log(1-z\overline{\zeta}\omega^{2k})\Big]\Bigg\},$$

*where $\Delta_1$, $B_0$ are defined by* (4.29) *and* (4.33) *respectively.*

**Proof:** We observe that

$$-\frac{1}{2\pi i}\int\limits_L \Delta_1(\widetilde{\zeta},\zeta)\sum_{k=0}^{n-1}2\log\left|(\widetilde{\zeta}-z\omega^{2k})(\overline{\widetilde{\zeta}}-z\omega^{2k})\right|^2\frac{\mathrm{d}\widetilde{\zeta}}{\widetilde{\zeta}}$$
$$=-2\sum_{k=0}^{n-1}\frac{1}{2\pi i}\int\limits_{|\widetilde{\zeta}|=1}\left(\frac{\widetilde{\zeta}}{\zeta}+\widetilde{\zeta}\overline{\widetilde{\zeta}}-2\right)\log|1-\overline{\zeta}\widetilde{\zeta}|^2\log\left|(1-z\widetilde{\zeta}\omega^{2k})(1-z\overline{\widetilde{\zeta}}\omega^{2k})\right|^2\frac{\mathrm{d}\widetilde{\zeta}}{\widetilde{\zeta}}$$
$$=\sum_{k=0}^{n-1}2(J_1+J_2+J_3+J_4),$$

where

$$J_1=-\frac{1}{2\pi i}\int\limits_{|\widetilde{\zeta}|=1}\left(\frac{\widetilde{\zeta}}{\zeta}+\widetilde{\zeta}\overline{\zeta}-2\right)\log(1-\overline{\zeta}\widetilde{\zeta})\log\Big[(1-\overline{z}\widetilde{\zeta}\omega^{-2k})(1-z\widetilde{\zeta}\omega^{2k})\Big]\frac{\mathrm{d}\widetilde{\zeta}}{\widetilde{\zeta}}=0,$$

$$J_2=\overline{-\frac{1}{2\pi i}\int\limits_{|\widetilde{\zeta}|=1}\left(\frac{\widetilde{\zeta}}{\zeta}+\widetilde{\zeta}\overline{\zeta}-2\right)\log(1-\zeta\overline{\widetilde{\zeta}})\log\Big[(1-\overline{z}\overline{\widetilde{\zeta}}\omega^{-2k})(1-z\overline{\widetilde{\zeta}}\omega^{2k})\Big]\frac{\mathrm{d}\widetilde{\zeta}}{\widetilde{\zeta}}}$$
$$=\overline{-\frac{1}{2\pi i}\int\limits_{|\widetilde{\zeta}|=1}\left[\left(\frac{1}{\overline{\zeta}}+\zeta-2\widetilde{\zeta}\right)\log(1-\overline{\zeta}\widetilde{\zeta})\log\Big[(1-\overline{z}\widetilde{\zeta}\omega^{-2k})(1-z\widetilde{\zeta}\omega^{2k})\Big]\right]\frac{\mathrm{d}\widetilde{\zeta}}{\widetilde{\zeta}^2}}$$
$$=\overline{-\partial_{\widetilde{\zeta}}\left[\left(\frac{1}{\overline{\zeta}}+\zeta-2\widetilde{\zeta}\right)\log(1-\overline{\zeta}\widetilde{\zeta})\log(1-\overline{z}\widetilde{\zeta}\omega^{-2k})(1-z\widetilde{\zeta}\omega^{2k})\right]_{\widetilde{\zeta}=0}}=0,$$

Moreover, from Lemma 4.2.2 and Lemma 4.2.3,

$$
\begin{aligned}
J_3 &= -\frac{1}{2\pi i}\int\limits_{|\widetilde{\zeta}|=1}\left(\frac{\widetilde{\zeta}}{\zeta}+\overline{\widetilde{\zeta}\zeta}-2\right)\log(1-\overline{\zeta}\widetilde{\zeta})\log\left[(1-z\overline{\overline{\widetilde{\zeta}}}\omega^{-2k})(1-\overline{z}\overline{\overline{\widetilde{\zeta}}}\omega^{-2k})\right]\frac{\mathrm{d}\widetilde{\zeta}}{\widetilde{\zeta}}\\
&= \sum_{l=0}^{\infty}\frac{\zeta^{l+1}}{(l+1)2\pi i}\int\limits_{|\widetilde{\zeta}|=1}\left[\left(\frac{1}{\overline{\overline{\zeta}}}+\zeta-2\widetilde{\zeta}\right)\log\left[(1-\overline{z}\widetilde{\zeta}\omega^{2k})(1-z\widetilde{\zeta}\omega^{2k})\right]\right]\frac{\mathrm{d}\widetilde{\zeta}}{\widetilde{\zeta}^{l+3}}\\
&= \sum_{l=0}^{\infty}\frac{\zeta^{l+1}}{(l+1)(l+2)!}\left[\left(\frac{1}{\overline{\overline{\zeta}}}+\zeta-2\widetilde{\zeta}\right)\log(1-\overline{z}\widetilde{\zeta}\omega^{2k})(1-z\widetilde{\zeta}\omega^{2k})\right]^{(l+2)}_{\widetilde{\zeta}=0}\\
&= \sum_{l=0}^{\infty}\frac{2}{(l+1)^2}\left[(\overline{z}\overline{\zeta}\omega^{-2k})^{l+1}+(z\overline{\zeta}\omega^{-2k})^{l+1}\right]-\frac{|\zeta|^2+1}{\zeta}(z+\overline{z})\omega^{-2k}\\
&\quad -\frac{|\zeta|^2+1}{|\zeta|^2}\left[(1-\overline{z}\overline{\zeta}\omega^{-2k})\log(1-\overline{z}\overline{\zeta}\omega^{-2k})+(1-z\overline{\zeta}\omega^{-2k})\log(1-z\overline{\zeta}\omega^{-2k})\right],
\end{aligned}
$$

and

$$
\begin{aligned}
J_4 &= -\frac{1}{2\pi i}\int\limits_{|\widetilde{\zeta}|=1}\left(\frac{\widetilde{\zeta}}{\zeta}+\overline{\widetilde{\zeta}\zeta}-2\right)\log(1-\zeta\overline{\overline{\widetilde{\zeta}}})\log\left[(1-z\widetilde{\zeta}\omega^{2k})(1-\overline{z}\widetilde{\zeta}\omega^{2k})\right]\frac{\mathrm{d}\widetilde{\zeta}}{\widetilde{\zeta}}\\
&= \sum_{l=0}^{\infty}\frac{\zeta^{l+1}}{(l+1)2\pi i}\int\limits_{|\widetilde{\zeta}|=1}\left[\left(\frac{\widetilde{\zeta}}{\zeta}+\overline{\zeta}\widetilde{\zeta}-2\right)\log(1-\overline{z}\widetilde{\zeta}\omega^{2k})(1-z\widetilde{\zeta}\omega^{2k})\right]\frac{\mathrm{d}\widetilde{\zeta}}{\widetilde{\zeta}^{l+2}}\\
&= \sum_{l=0}^{\infty}\frac{\zeta^{l+1}}{(l+1)(l+1)!}\left[\left(\frac{\widetilde{\zeta}}{\zeta}+\overline{\zeta}\widetilde{\zeta}-2\right)\log(1-\overline{z}\widetilde{\zeta}\omega^{2k})(1-z\widetilde{\zeta}\omega^{2k})\right]^{(l+1)}_{\widetilde{\zeta}=0}\\
&= \sum_{l=0}^{\infty}\frac{2}{(l+1)^2}\left[(\overline{z}\zeta\omega^{2k})^{l+1}+(z\zeta\omega^{2k})^{l+1}\right]-2\left(1+|\zeta|^2\right)\\
&\quad -(1+|\zeta|^2)\left[\frac{(1-\overline{z}\zeta\omega^{2k})\log(1-\overline{z}\zeta\omega^{2k})}{\overline{z}\zeta\omega^{2k}}+\frac{(1-z\zeta\omega^{2k})\log(1-z\zeta\omega^{2k})}{z\zeta\omega^{2k}}\right].
\end{aligned}
$$

Hence, the proof is completed.

**Remark 4.2.1.** Here the Taylor expansion of $\log(1-x)$ is used, that is,

$$\log(1-x) = -\sum_{l=0}^{\infty}\frac{x^{l+1}}{l+1},\qquad |x|<1.$$

Therefore, from (4.23), Theorem 4.2.1 and Lemmas 4.2.4-4.2.6,

$$\begin{aligned}
\widehat{h}_2(z,\zeta) &= -\sum_{k=0}^{n-1}\sum_{l=0}^{\infty}\frac{4}{(l+1)^2}\Big[(\overline{z}\zeta\omega^{-2k})^{l+1}+(z\zeta\omega^{2k})^{l+1} \\
&\qquad\qquad +(\overline{z}\overline{\zeta}\omega^{-2k})^{l+1}+(z\overline{\zeta}\omega^{2k})^{l+1}\Big] \\
&+\sum_{k=0}^{n-1}\Bigg\{8(|z|^2+|\zeta|^2)+\big(z\overline{\zeta}\omega^{2k}+\overline{z}\zeta\omega^{-2k}-z\overline{\zeta}-\overline{z}\zeta\big)\log|\zeta-z\omega^{2k}|^2 \\
&+\big(z\zeta\omega^{2k}+\overline{z}\overline{\zeta}\omega^{-2k}-z\overline{\zeta}-\overline{z}\zeta\big)\log|\zeta-\overline{z}\omega^{-2k}|^2 \\
&+\left(\frac{\omega^{-2k}}{z\overline{\zeta}}+\frac{\zeta}{z}\omega^{-2k}+\frac{\overline{z}}{\overline{\zeta}}\omega^{-2k}-z\overline{\zeta}\omega^{2k}-z\overline{\zeta}-\overline{z}\zeta\right)\log(1-z\overline{\zeta}\omega^{2k}) \\
&+\left(\frac{\omega^{2k}}{\overline{z}\zeta}+\frac{\overline{\zeta}}{\overline{z}}\omega^{2k}+\frac{z}{\zeta}\omega^{2k}-\overline{z}\zeta\omega^{-2k}-z\overline{\zeta}-\overline{z}\zeta\right)\log(1-\overline{z}\zeta\omega^{-2k}) \\
&+\left(\frac{\omega^{-2k}}{z\zeta}+\frac{\overline{\zeta}}{z}\omega^{-2k}+\frac{\overline{z}}{\zeta}\omega^{-2k}-z\zeta\omega^{2k}-z\overline{\zeta}-\overline{z}\zeta\right)\log(1-z\zeta\omega^{2k}) \\
&+\left(\frac{\omega^{2k}}{\overline{z}\overline{\zeta}}+\frac{\zeta}{\overline{z}}\omega^{2k}+\frac{z}{\overline{\zeta}}\omega^{2k}-\overline{z}\overline{\zeta}\omega^{-2k}-z\overline{\zeta}-\overline{z}\zeta\right)\log(1-\overline{z}\overline{\zeta}\omega^{-2k})\Bigg\}.
\end{aligned} \tag{4.36}$$

Finally, by (4.20) and (4.36), $N_2(z,\zeta)$ can be represented as

$$\begin{aligned}
N_2(z,\zeta) &= 4\sum_{k=0}^{n-1}\big(|\zeta-z\omega^{2k}|^2+|\zeta-\overline{z}\omega^{-2k}|^2\big) \\
&\quad -\sum_{k=0}^{n-1}\sum_{l=1}^{\infty}\frac{4}{(l+1)^2}\Big[(\overline{z}\zeta\omega^{-2k})^{l+1}+(z\zeta\omega^{2k})^{l+1} \\
&\qquad\qquad +(\overline{z}\overline{\zeta}\omega^{-2k})^{l+1}+(z\overline{\zeta}\omega^{2k})^{l+1}\Big] \\
&-\sum_{k=0}^{n-1}\Bigg\{\big|\zeta-z\omega^{2k}\big|^2\log\big|\zeta-z\omega^{2k}\big|^2+\big|\zeta-\overline{z}\omega^{-2k}\big|^2\log\big|\zeta-\overline{z}\omega^{-2k}\big|^2 \\
&\quad+\big|\zeta+z\omega^{2k}\big|^2\log\big|1-z\overline{\zeta}\omega^{2k}\big|^2+\big|\zeta+\overline{z}\omega^{-2k}\big|^2\log\big|1-z\zeta\omega^{2k}\big|^2 \\
&\quad-(|\zeta|^2+1)(|z|^2+1)\Bigg[\frac{\log(1-z\zeta\omega^{2k})}{z\zeta\omega^{2k}}+\frac{\log(1-\overline{z}\overline{\zeta}\omega^{-2k})}{\overline{z}\overline{\zeta}\omega^{-2k}} \\
&\qquad+\frac{\log(1-z\overline{\zeta}\omega^{2k})}{z\overline{\zeta}\omega^{2k}}+\frac{\log(1-\overline{z}\zeta\omega^{-2k})}{\overline{z}\zeta\omega^{-2k}}\Bigg]\Bigg\}.
\end{aligned} \tag{4.37}$$

By computation, we can confirm that expression (4.37) exactly satisfies

$$\partial_z \partial_{\overline{z}} N_2(z,\zeta) = N_1(z,\zeta) \ \text{ in } \ \Omega,$$

$$\partial_{\nu_z} N_2(z,\zeta) = \begin{cases} -4n(|\zeta|^2-1), & z \in L\backslash\{1,\omega\}, \\ 0, & z \in (0,1) \cup (\omega,0). \end{cases}$$

Moreover, the normalization condition for $N_2(z,\zeta)$ is also true,

$$\begin{aligned} \frac{1}{2\pi i} \int_L N_2(z,\zeta) \frac{\mathrm{d}z}{z} &= 4(|\zeta|^2+1) - \sum_{l=0}^{\infty} \frac{4}{(l+1)^2 2\pi i} \int_{|z|=1} [(z\zeta)^{l+1} + (\overline{z}\overline{\zeta})^{l+1}] \frac{\mathrm{d}z}{z} \\ &- \frac{2(|\zeta|^2+1)}{2\pi i} \int_{|z|=1} \left\{ \log|\zeta - z|^2 - \frac{\log(1-z\zeta)}{z\zeta} - \frac{\log(1-\overline{z}\overline{\zeta})}{\overline{z}\overline{\zeta}} \right\} \frac{\mathrm{d}z}{z} = 0. \end{aligned}$$

Thus, all the above verifications mean that expression (4.37) is the desired biharmonic Neumann function.

**Remark 4.2.2.** When $n = 1$, the biharmonic Neumann function for $\mathbb{D}^+$ is

$$\begin{aligned} N_2(z,\zeta) &= 8(|z|^2+|\zeta|^2) - |\zeta - z|^2 \log|\zeta - z|^2 - |\zeta - \overline{z}|^2 \log|\zeta - \overline{z}|^2 \\ &\quad - |\zeta + z|^2 \log\left|1 - z\overline{\zeta}\right|^2 - |\zeta + \overline{z}|^2 \log|1 - z\zeta|^2 \\ &\quad - \sum_{l=0}^{\infty} \frac{4}{(l+1)^2} \left[(\overline{z}\zeta)^{l+1} + (z\zeta)^{l+1} + (\overline{z}\overline{\zeta})^{l+1} + (z\overline{\zeta})^{l+1}\right] \\ &\quad + (|\zeta|^2+1)(|z|^2+1) \\ &\quad \times \left[ \frac{\log(1-z\zeta)}{z\zeta} + \frac{\log(1-\overline{z}\overline{\zeta})}{\overline{z}\overline{\zeta}} + \frac{\log(1-z\overline{\zeta})}{z\overline{\zeta}} + \frac{\log(1-\overline{z}\zeta)}{\overline{z}\zeta} \right]. \end{aligned}$$

## 4.3 Dirichlet and Neumann Problems for the Bi-Poisson Equation

On the basis of the biharmonic Green and the biharmonic Neumann functions, we discuss the related Dirichlet and Neumann problems.

Firstly, the biharmonic Green function provides a representation formula as follows.

**Theorem 4.3.1.** [6, 27] *Any* $w \in C^4(\Omega;\mathbb{C}) \cap C^3(\overline{\Omega};\mathbb{C})$ *can be expressed as*

$$w(z) = -\frac{1}{4\pi}\int\limits_{\partial\Omega} \partial_{\nu_\zeta} G_1(z,\zeta) w(\zeta) \mathrm{d}s_\zeta - \frac{1}{4\pi}\int\limits_{\partial\Omega} \partial_{\nu_\zeta} \widehat{G}_2(z,\zeta)\partial_\zeta\partial_{\overline{\zeta}} w(\zeta)\mathrm{d}s_\zeta$$
$$-\frac{1}{\pi}\int\limits_{\Omega} \widehat{G}_2(z,\zeta)(\partial_\zeta\partial_{\overline{\zeta}})^2 w(\zeta)\mathrm{d}\xi\mathrm{d}\eta,$$

*where* $G_1(z,\zeta)$, $\widehat{G}_2(z,\zeta)$ *are the harmonic Green and the biharmonic Green functions, respectively, for the domain* $\Omega$.

Similarly to the result in [27], the above representation formula also gives a solution to the Dirichlet problem for the bi-Poisson equation.

**Theorem 4.3.2.** *The Dirichlet problem*

$$\begin{cases} (\partial_z\partial_{\overline{z}})^2 w(z) = f & \text{in } \Omega, \\ w = \gamma_0, \quad \partial_z\partial_{\overline{z}} w = \gamma_1 & \text{on } \partial\Omega, \end{cases}$$

*with* $f \in L_p(\Omega;\mathbb{C})$, $p > 2$, $\gamma_0$, $\gamma_1 \in C(\partial\Omega;\mathbb{C})$ *is uniquely solvable. The solution is represented as*

$$w(z) = -\frac{1}{4\pi}\int\limits_{\partial\Omega}\left[\partial_{\nu_\zeta} G_1(z,\zeta)\gamma_0(\zeta) + \partial_{\nu_\zeta}\widehat{G}_2(z,\zeta)\gamma_1(\zeta)\right]\mathrm{d}s_\zeta - \frac{1}{\pi}\int\limits_{\Omega}\widehat{G}_2(z,\zeta) f(\zeta)\mathrm{d}\xi\mathrm{d}\eta,$$

*with* $G_1(z,\zeta)$, $\widehat{G}_2(z,\zeta)$ *given in* (4.1), (4.16) *respectively.*

Finally, we have the following representation formula related to the Neumann problem for the bi-Poisson equation.

**Theorem 4.3.3.** *Any* $w \in C^4(\Omega;\mathbb{C}) \cap C^3(\overline{\Omega};\mathbb{C})$ *can be represented as*

$$\begin{aligned} w(z) &= \frac{n}{\pi i}\int\limits_{L} w(\zeta)\frac{\mathrm{d}\zeta}{\zeta} + \frac{n(|z|^2-1)}{\pi i}\int\limits_{L}\partial_\zeta\partial_{\overline{\zeta}} w(\zeta)\frac{\mathrm{d}\zeta}{\zeta} \\ &+\frac{1}{4\pi}\int\limits_{\partial\Omega}\left[N_1(z,\zeta)\partial_{\nu_\zeta} w(\zeta) + N_2(z,\zeta)\partial_{\nu_\zeta}(\partial_\zeta\partial_{\overline{\zeta}})w(\zeta)\right]\mathrm{d}s_\zeta \\ &-\frac{1}{\pi}\int\limits_{\Omega} N_2(z,\zeta)(\partial_\zeta\partial_{\overline{\zeta}})^2 w(\zeta)\mathrm{d}\xi\mathrm{d}\eta, \quad z \in \Omega, \end{aligned}$$

*where* $N_1(z,\zeta), N_2(z,\zeta)$ *are the harmonic Neumann and the biharmonic Neumann functions for* $\Omega$ *respectively.*

**Proof:** Applying the representation formula in Theorem 3.2.1 to $\partial_z\partial_{\overline{z}}w(z)$ leads to

$$\begin{aligned}\partial_z\partial_{\overline{z}}w(z) &= \frac{1}{4\pi}\int\limits_{\partial\Omega}\left[\partial_{\nu_\zeta}\partial_\zeta\partial_{\overline{\zeta}}w(\zeta)N_1(z,\zeta)-\partial_\zeta\partial_{\overline{\zeta}}w(\zeta)\partial_{\nu_\zeta}N_1(z,\zeta)\right]\mathrm{d}s_\zeta \\ &\quad -\frac{1}{\pi}\int\limits_{\Omega}N_1(z,\zeta)(\partial_\zeta\partial_{\overline{\zeta}})^2w(\zeta)\mathrm{d}\xi\mathrm{d}\eta,\end{aligned} \tag{4.38}$$

In addition, we also have

$$\begin{aligned}w(z) = \frac{1}{4\pi}\int\limits_{\partial\Omega}\left[\partial_{\nu_\zeta}w(\zeta)N_1(z,\zeta)-w(\zeta)\partial_{\nu_\zeta}N_1(z,\zeta)\right]\mathrm{d}s_\zeta \\ -\frac{1}{\pi}\int\limits_{\Omega}N_1(z,\zeta)\partial_\zeta\partial_{\overline{\zeta}}w(\zeta)\mathrm{d}\xi\mathrm{d}\eta,\end{aligned} \tag{4.39}$$

Then, putting (4.38) into the area integral of (4.39), we obtain the desired conclusion from the boundary behavior of $N_2(z,\zeta)$.

**Theorem 4.3.4.** *The Neumann problem for the bi-Poisson equation*

$$(\partial_z\partial_{\overline{z}})^2w=f \quad in \quad \Omega, \quad \partial_{\nu_z}w=\gamma_1, \quad \partial_{\nu_z}\partial_z\partial_{\overline{z}}w=\gamma_2 \quad on \quad \partial\Omega,$$
$$\frac{n}{\pi i}\int\limits_L w(\zeta)\frac{\mathrm{d}\zeta}{\zeta}=c_0, \quad \frac{n}{\pi i}\int\limits_L \partial_\zeta\partial_{\overline{\zeta}}w(\zeta)\frac{\mathrm{d}\zeta}{\zeta}=c_1,$$

*for* $f\in L_p(\Omega;\mathbb{C}),\ p>2,\ \gamma_1,\ \gamma_2\in C(\partial\Omega;\mathbb{C})$ *is solvable if and only if*

$$\frac{1}{2\pi}\int\limits_{\partial\Omega}\gamma_2(\zeta)\mathrm{d}s_\zeta=\frac{2}{\pi}\int\limits_\Omega f(\zeta)\mathrm{d}\xi\mathrm{d}\eta, \tag{4.40}$$

*and*

$$\frac{n}{\pi}\int\limits_{\partial\Omega}\gamma_1(\zeta)\mathrm{d}s_\zeta+\frac{n}{\pi}\int\limits_{\partial\Omega}(|\zeta|^2-1)\gamma_2(\zeta)\mathrm{d}s_\zeta=2c_1+\frac{4n}{\pi}\int\limits_\Omega(|\zeta|^2-1)f(\zeta)\mathrm{d}\xi\mathrm{d}\eta. \tag{4.41}$$

*The solution is uniquely expressed as*

$$\begin{aligned}w(z)=c_0+(|z|^2-1)c_1+\frac{1}{4\pi}\int\limits_{\partial\Omega}\left[N_1(z,\zeta)\gamma_1(\zeta)+N_2(z,\zeta)\gamma_2(\zeta)\right]\mathrm{d}s_\zeta \\ -\frac{1}{\pi}\int\limits_\Omega N_2(z,\zeta)f(\zeta)\mathrm{d}\xi\mathrm{d}\eta, \quad z\in\Omega,\end{aligned} \tag{4.42}$$

*with* $N_1(z,\zeta),\ N_2(z,\zeta)$ *given by* (4.17), (4.37) *respectively.*

**Proof:** The boundary problem above can be transformed into two boundary problems,

$$\partial_z \partial_{\overline{z}} w = w_1 \ \text{ in } \ \Omega, \ \ \partial_{\nu_z} w = \gamma_1 \ \text{ on } \ \partial\Omega, \ \ \frac{n}{\pi i} \int_L w(\zeta) \frac{\mathrm{d}\zeta}{\zeta} = c_0,$$

and

$$\partial_z \partial_{\overline{z}} w_1 = f \ \text{ in } \ \Omega, \ \ \partial_{\nu_z} w_1 = \gamma_2 \ \text{ on } \ \partial\Omega, \ \ \frac{n}{\pi i} \int_L w_1(\zeta) \frac{\mathrm{d}\zeta}{\zeta} = c_1.$$

Then from Lemma 4.2.1, if and only if

$$\frac{1}{2\pi} \int_{\partial\Omega} \gamma_2(\zeta) \mathrm{d}s_\zeta = \frac{2}{\pi} \int_\Omega f(\zeta) \mathrm{d}\xi \mathrm{d}\eta, \tag{4.43}$$

the second problem has the unique solution

$$w_1(z) = c_1 + \frac{1}{4\pi} \int_{\partial\Omega} N_1(z,\zeta)\gamma_2(\zeta) \mathrm{d}s_\zeta - \frac{1}{\pi} \int_\Omega N_1(z,\zeta) f(\zeta) \mathrm{d}\xi \mathrm{d}\eta. \tag{4.44}$$

Similarly, if and only if

$$\frac{1}{2\pi} \int_{\partial\Omega} \gamma_1(\zeta) \mathrm{d}s_\zeta = \frac{2}{\pi} \int_\Omega w_1(\zeta) \mathrm{d}\xi \mathrm{d}\eta, \tag{4.45}$$

is satisfied,

$$w(z) = c_0 + \frac{1}{4\pi} \int_{\partial\Omega} N_1(z,\zeta)\gamma_1(\zeta) \mathrm{d}s_\zeta - \frac{1}{\pi} \int_\Omega N_1(z,\zeta) w_1(\zeta) \mathrm{d}\xi \mathrm{d}\eta. \tag{4.46}$$

Putting (4.44) into (4.45) and (4.46) respectively, we obviously see that (4.41) and (4.42) follow from

$$\frac{1}{\pi} \int_\Omega 1 \mathrm{d}\xi \mathrm{d}\eta = \frac{1}{2n} \quad \text{and} \quad -\frac{1}{\pi} \int_\Omega N_1(z,\zeta) \mathrm{d}\xi \mathrm{d}\eta = |z|^2 - 1, \ \ z \in \overline{\Omega}.$$

This completes the proof.

# Chapter 5

# Triharmonic Boundary Value Problems for the Tri-Poisson Equation

In this Chapter, we continue to consider the construction of a triharmonic Green function and a triharmonic Neumann function explicitly in the fan-shaped domain with angle $\pi/n$ $(n \in \mathbb{N})$. A triharmonic Green function is established by convoluting the harmonic Green function with the above biharmonic Green function. At the same time, convoluting the harmonic Neumann function with the biharmonic Neumann function also results in a triharmonic Neumann function. Here $\Omega,\ L,\ \omega,\ \theta$ are defined as before.

## 5.1 Triharmonic Green Function

Firstly, convoluting the harmonic Green function with the above biharmonic Green function leads to

$$\widehat{G}_3(z,\zeta) = -\frac{1}{\pi}\int\limits_{\Omega} G_1(z,\widetilde{\zeta})\widehat{G}_2(\widetilde{\zeta},\zeta)d\widetilde{\xi}d\widetilde{\eta}, \quad z,\ \zeta \in \Omega, \tag{5.1}$$

where $G_1(z,\zeta),\ \widehat{G}_2(z,\zeta)$ are given in (4.1) and (4.16) respectively. Then we easily obtain that for any fixed $\zeta \in \Omega$, $\widehat{G}_3(z,\zeta)$ has the properties,

1. $\widehat{G}_3(z,\zeta)$ is triharmonic in $\Omega\backslash\{\zeta\}$,
2. $\widehat{G}_3(z,\zeta) + \frac{1}{4}|\zeta - z|^4 \log|\zeta - z|^2$ is triharmonic in $\Omega$,
3. $(\partial_z\partial_{\bar{z}})^2\widehat{G}_3(z,\zeta) = G_1(z,\zeta)$ for $z \in \Omega$,
$\widehat{G}_3(z,\zeta) = 0,\ \ \partial_z\partial_{\bar{z}}\widehat{G}_3(z,\zeta) = 0,\ \ (\partial_z\partial_{\bar{z}})^2\widehat{G}_3(z,\zeta) = 0$ for $z \in \partial\Omega$,
4. $\widehat{G}_3(z,\zeta) = \widehat{G}_3(\zeta,z)$ for $z \neq \zeta$.

Thus, we can rewrite $\widehat{G}_3(z,\zeta)$ as

$$\widehat{G}_3(z,\zeta) = \frac{1}{4}|\zeta - z|^4 G_1(z,\zeta) + h_3(z,\zeta), \tag{5.2}$$

where $h_3(z,\zeta)$ is a triharmonic function in $\Omega$. Hence, $h_3(z,\zeta)$ is the solution to the Dirichlet problem

$$\begin{cases} (\partial_z\partial_{\bar z})^2 h_3(z,\zeta) = f_1(z,\zeta) \ \text{ in } \ \Omega, \\ h_3(z,\zeta)=0, \ \partial_z\partial_{\bar z} h_3(z,\zeta) = g_1(z,\zeta) \ \text{ for } z\in\partial\Omega, \end{cases} \tag{5.3}$$

with

$$\begin{aligned} f_1(z,\zeta) &= 2(\overline{\zeta}-\overline{z})\partial_{\bar z}G_1 + 2(\zeta - z)\partial_z G_1 - \frac{1}{2}(\overline{\zeta}-\overline{z})^2\partial_{\bar z}^2 G_1 - \frac{1}{2}(\zeta-z)^2\partial_z^2 G_1 \\ &= \sum_{k=0}^{n-1}\mathrm{Re}\Bigg\{\frac{(|\zeta|^2-\omega^{-2k})^2}{(\omega^{-2k}-z\overline{\zeta})^2} + \frac{(\overline{\zeta}-\zeta\omega^{2k})^2}{(\overline{\zeta}-z\omega^{2k})^2} - \frac{(\zeta^2-\omega^{-2k})^2}{(\omega^{-2k}-z\zeta)^2} - \frac{(\zeta-\zeta\omega^{2k})^2}{(\zeta-z\omega^{2k})^2} \\ &\quad +2\left[\frac{\zeta^2-\omega^{-2k}}{\omega^{-2k}-z\zeta} + \frac{\zeta\omega^{2k}-\zeta}{\zeta-z\omega^{2k}} - \frac{|\zeta|^2-\omega^{-2k}}{\omega^{-2k}-z\overline{\zeta}} - \frac{\zeta\omega^{2k}-\overline{\zeta}}{\overline{\zeta}-z\omega^{2k}}\right]\Bigg\}, \end{aligned} \tag{5.4}$$

and

$$\begin{aligned} g_1(z,\zeta) &= \frac{1}{2}(\zeta-z)(\overline{\zeta}-\overline{z})^2\partial_{\bar z}G_1 + \frac{1}{2}(\zeta-z)^2(\overline{\zeta}-\overline{z})\partial_z G_1 \\ &= |\zeta-z|^2\sum_{k=0}^{n-1}\mathrm{Re}\left\{\frac{\zeta^2-\omega^{-2k}}{\omega^{-2k}-z\zeta} + \frac{\zeta\omega^{2k}-\zeta}{\zeta-z\omega^{2k}} - \frac{|\zeta|^2-\omega^{-2k}}{\omega^{-2k}-z\overline{\zeta}} - \frac{\zeta\omega^{2k}-\overline{\zeta}}{\overline{\zeta}-z\omega^{2k}}\right\}. \end{aligned} \tag{5.5}$$

Next, we need a new triharmonic function,

$$\begin{aligned} F_1(z,\zeta) \ &= -\frac{1}{4}(z^2\overline{\zeta}^2 + \overline{z}^2\zeta^2)G_1(z,\zeta) + \frac{1}{2}(|z|^2+|\zeta|^2)(\overline{z}\zeta + z\overline{\zeta})G_1(z,\zeta) \\ &\quad + \sum_{k=0}^{n-1}\Bigg\{\Bigg[\frac{|z|^2+|\zeta|^2}{2|\zeta|^2}(\overline{z}\overline{\zeta}\omega^{-2k} + z\zeta\omega^{2k}) \\ &\qquad\qquad - \frac{z^2\zeta^2\omega^{4k} + \overline{z}^2\overline{\zeta}^2\omega^{-4k}}{4|\zeta|^4}\Bigg]\log|\omega^{-2k}-z\zeta|^2 \\ &\quad - \Bigg[\frac{|z|^2+|\zeta|^2}{2|\zeta|^2}(\overline{z}\zeta\omega^{-2k} + z\overline{\zeta}\omega^{2k}) \\ &\qquad\qquad - \frac{\overline{z}^2\zeta^2\omega^{-4k} + z^2\overline{\zeta}^2\omega^{4k}}{4|\zeta|^4}\Bigg]\log|\omega^{-2k}-z\overline{\zeta}|^2 \\ &\quad + \Bigg[\frac{1}{2}(|z|^2+|\zeta|^2)(\overline{z}\zeta\omega^{-2k} + z\overline{\zeta}\omega^{2k}) \\ &\qquad\qquad - \frac{1}{4}(z^2\overline{\zeta}^2\omega^{4k} + \overline{z}^2\zeta^2\omega^{-4k})\Bigg]\log|\zeta - z\omega^{2k}|^2 \end{aligned}$$

$$-\left[\frac{1}{2}(|z|^2+|\zeta|^2)(\overline{z}\overline{\zeta}\omega^{-2k}+z\zeta\omega^{2k}) \right.$$
$$\left. -\frac{1}{4}(z^2\zeta^2\omega^{4k}+\overline{z}^2\overline{\zeta}^2\omega^{-4k})\right]\log|\overline{\zeta}-z\omega^{2k}|^2\Bigg\}, \quad z,\ \zeta\in\Omega. \tag{5.6}$$

Then we have the following result.

**Theorem 5.1.1.** *For $z,\ \zeta\in\Omega$,*

$$h_3(z,\zeta)=F_1(z,\zeta)\quad+\frac{1}{4\pi i}\int\limits_L\Bigg\{\partial_{\nu_{\widetilde{\zeta}}}G_1(z,\widetilde{\zeta})F_1(\widetilde{\zeta},\zeta) +\partial_{\nu_{\widetilde{\zeta}}}\widehat{G}_2(z,\widetilde{\zeta})\left[\partial_{\widetilde{\zeta}\overline{\widetilde{\zeta}}}F_1(\widetilde{\zeta},\zeta)-g_1(\widetilde{\zeta},\zeta)\right]\Bigg\}\frac{d\widetilde{\zeta}}{\widetilde{\zeta}}. \tag{5.7}$$

*Here $F_1,\ g_1$ are given in* (5.6) *and* (5.5)*, respectively, and for $\widetilde{\zeta}\in L$,*

$$\partial_{\nu_{\widetilde{\zeta}}}G_1(z,\widetilde{\zeta})=2\sum_{k=0}^{n-1}\left[\frac{1}{1-z\widetilde{\zeta}\omega^{2k}}-\frac{1}{1-\overline{z}\widetilde{\zeta}\omega^{-2k}}+\frac{\widetilde{\zeta}}{\widetilde{\zeta}-\overline{z}\omega^{-2k}}-\frac{\widetilde{\zeta}}{\widetilde{\zeta}-z\omega^{2k}}\right], \tag{5.8}$$

*and*

$$\begin{aligned}\partial_{\nu_{\widetilde{\zeta}}}\widehat{G}_2(z,\widetilde{\zeta}) &= 2(|z|^2-1)\sum_{k=0}^{n-1}\Bigg\{\frac{\widetilde{\zeta}\omega^{2k}}{z}\log(1-z\overline{\widetilde{\zeta}}\omega^{-2k}) \\ &+\frac{\overline{\widetilde{\zeta}}\omega^{-2k}}{\overline{z}}\log(1-\overline{z}\widetilde{\zeta}\omega^{2k})-\frac{\overline{\widetilde{\zeta}}\omega^{-2k}}{z}\log(1-z\widetilde{\zeta}\omega^{2k}) \\ &-\frac{\widetilde{\zeta}\omega^{2k}}{\overline{z}}\log(1-\overline{z}\overline{\widetilde{\zeta}}\omega^{-2k})\Bigg\}.\end{aligned} \tag{5.9}$$

**Proof:** We can verify that

$$\begin{cases}(\partial_z\partial_{\overline{z}})^2F_1(z,\zeta)=f_1(z,\zeta), & z,\ \zeta\in\Omega,\\ F_1(z,\zeta)=0, & z\in[0,1]\cup[\omega,0].\end{cases} \tag{5.10}$$

Then, by Theorem 4.3.2,

$$\begin{aligned}&-\frac{1}{\pi}\int\limits_{\Omega}\widehat{G}_2(z,\widetilde{\zeta})f_1(\widetilde{\zeta},\zeta)d\widetilde{\xi}d\widetilde{\eta}\\ &=F_1(z,\zeta)+\frac{1}{4\pi}\int\limits_{\partial\Omega}\left[\partial_{\nu_{\widetilde{\zeta}}}G_1(z,\widetilde{\zeta})F_1(\widetilde{\zeta},\zeta)+\partial_{\nu_{\widetilde{\zeta}}}\widehat{G}_2(z,\widetilde{\zeta})\partial_{\widetilde{\zeta}\overline{\widetilde{\zeta}}}F_1(\widetilde{\zeta},\zeta)\right]ds_{\widetilde{\zeta}}.\end{aligned} \tag{5.11}$$

Moreover, by (5.3) and Theorem 4.3.2,

$$h_3(z,\zeta) = -\frac{1}{4\pi}\int\limits_{\partial\Omega} \partial_{\nu_{\widetilde{\zeta}}}\widehat{G}_2(z,\widetilde{\zeta})g_1(\widetilde{\zeta},\zeta)\mathrm{d}s_{\widetilde{\zeta}} - \frac{1}{\pi}\int\limits_{\Omega}\widehat{G}_2(z,\widetilde{\zeta})f_1(\widetilde{\zeta},\zeta)\mathrm{d}\widetilde{\xi}\mathrm{d}\widetilde{\eta}. \tag{5.12}$$

Hence, from (5.11) and (5.12),

$$\begin{aligned} h_3(z,\zeta) = F_1(z,\zeta) \quad &+\frac{1}{4\pi}\int\limits_{\partial\Omega}\Bigg\{\partial_{\nu_{\widetilde{\zeta}}}G_1(z,\widetilde{\zeta})F_1(\widetilde{\zeta},\zeta) \\ &+\partial_{\nu_{\widetilde{\zeta}}}\widehat{G}_2(z,\widetilde{\zeta})\left[\partial_{\widetilde{\zeta}\overline{\widetilde{\zeta}}}F_1(\widetilde{\zeta},\zeta) - g_1(\widetilde{\zeta},\zeta)\right]\Bigg\}\mathrm{d}s_{\widetilde{\zeta}}. \end{aligned} \tag{5.13}$$

We also know that for $z \in [0,1]$,

$$\begin{aligned} \partial_z\partial_{\overline{z}}F_1(z,\zeta) \quad &= \frac{1}{2}(\overline{\zeta}-\zeta)|\zeta - z|^2\sum_{k=0}^{n-1}\Bigg[\frac{\overline{\zeta}}{\omega^{-2k} - z\overline{\zeta}} - \frac{\zeta}{\omega^{-2k} - z\zeta} \\ &\qquad + \frac{\omega^{2k}}{\overline{\zeta} - z\omega^{2k}} - \frac{\omega^{2k}}{\zeta - z\omega^{2k}}\Bigg], \end{aligned} \tag{5.14}$$

and for $z \in [\omega, 0]$,

$$\begin{aligned} \partial_z\partial_{\overline{z}}F_1(z,\zeta) \quad &= \frac{1}{2}\left[(\zeta - z)^2(\overline{\zeta} - \overline{z}) - (\zeta - z)(\overline{\zeta} - \overline{z})^2 e^{2\theta i}\right] \\ &\times\sum_{k=0}^{n-1}\left[\frac{\zeta}{\omega^{-2k} - z\zeta} + \frac{\omega^{2k}}{\zeta - z\omega^{2k}} - \frac{\overline{\zeta}}{\omega^{-2k} - z\overline{\zeta}} - \frac{\omega^{2k}}{\overline{\zeta} - z\omega^{2k}}\right]. \end{aligned} \tag{5.15}$$

Thus, by (5.5), (5.14) and (5.15), we obtain,

$$\partial_{\widetilde{\zeta}\overline{\widetilde{\zeta}}}F_1(\widetilde{\zeta},\zeta) - g_1(\widetilde{\zeta},\zeta) = 0 \quad \text{for} \quad \widetilde{\zeta} \in [0,1]\cup[\omega,0]. \tag{5.16}$$

Also from (5.10), (5.13) and (5.16), the conclusion is true.

From (5.6) and (5.5), for $z \in L$,

$$\begin{aligned} F_1(z,\zeta) = (|\zeta|^4 - 1) \quad &\sum_{k=0}^{n-1}\mathrm{Re}\Bigg\{\left[\frac{\overline{z}^2\omega^{-4k}}{2\zeta^2} - \frac{\overline{z}\omega^{-2k}}{\zeta}\right]\log|\zeta - \overline{z}\omega^{-2k}|^2 \\ &-\left[\frac{z^2\omega^{4k}}{2\zeta^2} - \frac{z\omega^{2k}}{\zeta}\right]\log|\zeta - z\omega^{2k}|^2\Bigg\}, \end{aligned} \tag{5.17}$$

and

$$\partial_z\partial_{\overline{z}}F_1(z,\zeta)-g_1(z,\zeta)=2(|\zeta|^2-1)\sum_{k=0}^{n-1}\mathrm{Re}\left\{\Delta(z\omega^{2k},\zeta)-\triangle(\overline{z}\omega^{-2k},\zeta)\right\}, \quad (5.18)$$

with

$$\Delta(z,\zeta)=\frac{z}{\zeta}\log|\zeta-z|^2+\frac{1}{2(\zeta-z)}\left[\frac{\overline{z}\zeta}{\overline{\zeta}}-\frac{z^2}{\zeta}+2z\right]. \quad (5.19)$$

Then, the next two lemmas are true.

**Lemma 5.1.1.** *For $z,\ \zeta\in\Omega$,*

$$\begin{aligned}
&K_1(z,\zeta)=\frac{1}{4\pi i}\int\limits_L \partial_{\nu_{\widetilde{\zeta}}}G_1(z,\widetilde{\zeta})F_1(\widetilde{\zeta},\zeta)\frac{d\widetilde{\zeta}}{\widetilde{\zeta}}\\
&=\frac{(|\zeta|^4-1)(|z|^2-1)}{4|z|^2|\zeta|^2}(z-\overline{z})(\zeta-\overline{\zeta})B_0\\
&\quad-\frac{|\zeta|^4-1}{4}\sum_{k=0}^{n-1}\Bigg\{\left[\frac{(\omega^{-2k}-z\zeta)^2}{z^2\zeta^2}+\frac{(\overline{\zeta}-z\omega^{2k})^2}{\overline{\zeta}^2}-2\right]\log(1-z\zeta\omega^{2k})\\
&\quad+\left[\frac{(\omega^{2k}-\overline{z}\overline{\zeta})^2}{\overline{z}^2\overline{\zeta}^2}+\frac{(\zeta-\overline{z}\omega^{-2k})^2}{\zeta^2}-2\right]\log(1-\overline{z}\overline{\zeta}\omega^{-2k})\\
&\quad-\left[\frac{(\omega^{-2k}-z\overline{\zeta})^2}{z^2\overline{\zeta}^2}+\frac{(\zeta-z\omega^{2k})^2}{\zeta^2}-2\right]\log(1-z\overline{\zeta}\omega^{2k})\\
&\quad-\left[\frac{(\omega^{2k}-\overline{z}\zeta)^2}{\overline{z}^2\zeta^2}+\frac{(\overline{\zeta}-\overline{z}\omega^{-2k})^2}{\overline{\zeta}^2}-2\right]\log(1-\overline{z}\zeta\omega^{-2k})\Bigg\},
\end{aligned}$$

*with $\partial_{\nu_{\widetilde{\zeta}}}G_1(z,\widetilde{\zeta})$, $B_0$ given in* (5.8) *and* (4.33) *respectively.*

**Proof:** By (5.17) and (5.8), we can transform $K_1(z,\zeta)$ into an integral on the whole unit circle. That is

$$\begin{aligned}
K_1(z,\zeta)\ &=\frac{1}{4\pi i}\int\limits_L \partial_{\nu_{\widetilde{\zeta}}}G_1(z,\widetilde{\zeta})F_1(\widetilde{\zeta},\zeta)\frac{d\widetilde{\zeta}}{\widetilde{\zeta}}\\
&=\frac{1-|\zeta|^4}{2}\mathrm{Re}\left\{\frac{1}{\zeta^2 4\pi i}\int\limits_{|\widetilde{\zeta}|=1}\partial_{\nu_{\widetilde{\zeta}}}G_1(z,\widetilde{\zeta})(\widetilde{\zeta}-2\zeta)\log|\zeta-\widetilde{\zeta}|^2 d\widetilde{\zeta}\right\}.
\end{aligned}$$

**Lemma 5.1.2.** *For* $z,\ \zeta\in\Omega$,

$$\begin{aligned}
K_2(z,\zeta) &= \frac{1}{4\pi i}\int\limits_L \partial_{\nu_{\widetilde{\zeta}}}\widehat{G}_2(z,\widetilde{\zeta})\left[\partial_{\widetilde{\zeta}\overline{\widetilde{\zeta}}}F_1(\widetilde{\zeta},\zeta)-g_1(\widetilde{\zeta},\zeta)\right]\frac{\mathrm{d}\widetilde{\zeta}}{\widetilde{\zeta}}\\
&= \frac{(|\zeta|^2-1)(|z|^2-1)}{4|\zeta|^2}(z-\overline{z})(\zeta-\overline{\zeta})B_0+(|\zeta|^2-1)(|z|^2-1)\\
&\quad\times\sum_{k=0}^{n-1}\sum_{l=0}^{\infty}\frac{1}{(l+1)^2}\left[(z\overline{\zeta}\omega^{2k})^l+(\overline{z}\zeta\omega^{-2k})^l-(z\zeta\omega^{2k})^l-(\overline{z}\overline{\zeta}\omega^{-2k})^l\right]\\
&\quad+\frac{(|\zeta|^2-1)(|z|^2-1)}{2|\zeta|^2}\sum_{k=0}^{n-1}\Bigg\{\left(\frac{\overline{\zeta}}{z}\omega^{-2k}+z\zeta\omega^{2k}\right)\log(1-z\zeta\omega^{2k})\\
&\qquad+\left(\frac{\zeta}{\overline{z}}\omega^{2k}+\overline{z}\overline{\zeta}\omega^{-2k}\right)\log(1-\overline{z}\overline{\zeta}\omega^{-2k})\\
&\qquad-\left(\frac{\overline{\zeta}}{\overline{z}}\omega^{2k}+\overline{z}\zeta\omega^{-2k}\right)\log(1-\overline{z}\zeta\omega^{-2k})\\
&\qquad-\left(\frac{\zeta}{z}\omega^{-2k}+z\overline{\zeta}\omega^{2k}\right)\log(1-z\overline{\zeta}\omega^{2k})\Bigg\},
\end{aligned}$$

*where* $\partial_{\nu_{\widetilde{\zeta}}}\widehat{G}_2(z,\widetilde{\zeta}),\ B_0$ *are given in* (5.9) *and* (4.33) *respectively.*

**Proof:** Similarly, from (5.18) and (5.9), the lemma derives from

$$\begin{aligned}
K_2(z,\zeta) &= \frac{1}{4\pi i}\int\limits_L \partial_{\nu_{\widetilde{\zeta}}}\widehat{G}_2(z,\widetilde{\zeta})\left[\partial_{\widetilde{\zeta}\overline{\widetilde{\zeta}}}F_1(\widetilde{\zeta},\zeta)-g_1(\widetilde{\zeta},\zeta)\right]\frac{\mathrm{d}\widetilde{\zeta}}{\widetilde{\zeta}}\\
&= (|\zeta|^2-1)\mathrm{Re}\Bigg\{\frac{1}{2\pi i}\int\limits_{|\widetilde{\zeta}|=1}\partial_{\nu_{\widetilde{\zeta}}}\widehat{G}_2(z,\widetilde{\zeta})\Delta(\widetilde{\zeta},\zeta)\frac{\mathrm{d}\widetilde{\zeta}}{\widetilde{\zeta}}\Bigg\},
\end{aligned}$$

where $\Delta(\widetilde{\zeta},\zeta)$ is defined in (5.19).

Hence, from Theorem 5.1.1,

$$h_3(z,\zeta)=K_1(z,\zeta)+K_2(z,\zeta)+F_1(z,\zeta). \tag{5.20}$$

Furthermore, by (5.2), (5.20) and Lemmas 5.1.1-5.1.2, we see that $\widehat{G}_3(z,\zeta)$ can be expressed as

$$
\begin{aligned}
\widehat{G}_3(z,\zeta) = & -\frac{1}{4}(|\zeta|^2-1)(|z|^2-1)(z-\overline{z})(\zeta-\overline{\zeta})B_0 \\
& +(|\zeta|^2-1)(|z|^2-1)\sum_{k=0}^{n-1}\sum_{l=1}^{\infty}\frac{1}{(l+1)^2}\Big[(z\overline{\zeta}\omega^{2k})^l+(\overline{z}\zeta\omega^{-2k})^l \\
& \qquad -(z\zeta\omega^{2k})^l-(\overline{z}\overline{\zeta}\omega^{-2k})^l\Big] \\
& +\sum_{k=0}^{n-1}\Bigg\{\frac{1}{4}|\zeta-z\omega^{2k}|^4\log\left|\frac{1-z\overline{\zeta}\omega^{2k}}{\zeta-z\omega^{2k}}\right|^2+\frac{1}{4}|\zeta-\overline{z}\omega^{-2k}|^4\log\left|\frac{\zeta-\overline{z}\omega^{-2k}}{1-z\zeta\omega^{2k}}\right|^2 \\
& +\frac{(|\zeta|^4-1)(|z|^4-1)}{4}\Bigg[\frac{1}{z\zeta\omega^{2k}}+\frac{1}{\overline{z}\overline{\zeta}\omega^{-2k}}-\frac{1}{\overline{z}\zeta\omega^{-2k}}-\frac{1}{z\overline{\zeta}\omega^{2k}} \\
& +\frac{\log(1-z\zeta\omega^{2k})}{(z\zeta\omega^{2k})^2}+\frac{\log(1-\overline{z}\overline{\zeta}\omega^{-2k})}{(\overline{z}\overline{\zeta}\omega^{-2k})^2}-\frac{\log(1-\overline{z}\zeta\omega^{-2k})}{(\overline{z}\zeta\omega^{-2k})^2}-\frac{\log(1-z\overline{\zeta}\omega^{2k})}{(z\overline{\zeta}\omega^{2k})^2}\Bigg] \\
& -\frac{(|\zeta|^2-1)(|z|^2-1)(|z|^2+|\zeta|^2)}{2}\Bigg[\frac{\log(1-z\zeta\omega^{2k})}{z\zeta\omega^{2k}}+\frac{\log(1-\overline{z}\overline{\zeta}\omega^{-2k})}{\overline{z}\overline{\zeta}\omega^{-2k}} \\
& \qquad -\frac{\log(1-\overline{z}\zeta\omega^{-2k})}{\overline{z}\zeta\omega^{-2k}}-\frac{\log(1-z\overline{\zeta}\omega^{2k})}{z\overline{\zeta}\omega^{2k}}\Bigg]\Bigg\},
\end{aligned}
\tag{5.21}
$$

where $B_0$ is defined by (4.33). At the same time, we can validate that expression (5.21) is exactly the solution to the following problem

$$
\begin{cases}
(\partial_z\partial_{\overline{z}})^2\widehat{G}_3(z,\zeta)=G_1(z,\zeta) \quad \text{for} \quad z\in\Omega, \\
\widehat{G}_3(z,\zeta)=0, \quad \partial_z\partial_{\overline{z}}\widehat{G}_3(z,\zeta)=0 \quad \text{for} \quad z\in\partial\Omega.
\end{cases}
$$

That is to say, the expression (5.21) is just the desired triharmonic Green function for $\Omega$.

## 5.2 Triharmonic Neumann Function

Similarly, we consider the convolution of the harmonic Neumann function and the biharmonic Neumann function,

$$
N_3(z,\zeta)=-\frac{1}{\pi}\int_{\Omega}N_1(z,\widetilde{\zeta})N_2(\widetilde{\zeta},\zeta)\mathrm{d}\widetilde{\xi}\mathrm{d}\widetilde{\eta}, \quad z,\ \zeta\in\Omega, \tag{5.22}
$$

where $N_1(z,\zeta),\ N_2(z,\zeta)$ are given in (4.17) and (4.37) respectively.

For convenience, we restate some properties of $N_1(z,\zeta),\ N_2(z,\zeta)$,

$$\partial_{\nu_z} N_1(z,\zeta) = \begin{cases} -4n, & z \in L\backslash\{1,\omega\}, \\ 0, & z \in (0,1)\cup(\omega,0), \end{cases} \qquad \text{for } \ \zeta \in \Omega, \tag{5.23}$$

$$\partial_{\nu_z} N_2(z,\zeta) = \begin{cases} 4n(1-|\zeta|^2), & z \in L\backslash\{1,\omega\}, \\ 0, & z \in (0,1)\cup(\omega,0). \end{cases} \qquad \text{for } \ \zeta \in \Omega, \tag{5.24}$$

and the valid normalization conditions,

$$\int\limits_L N_1(z,\zeta)\frac{\mathrm{d}z}{z} = 0, \qquad \int\limits_L N_2(z,\zeta)\frac{\mathrm{d}z}{z} = 0. \tag{5.25}$$

Applying Theorem 4.3.4 to $w(z) = \frac{1}{4}(|z|^2-1)^2$, we have

$$-\frac{1}{\pi}\int\limits_\Omega N_2(z,\zeta)\mathrm{d}\xi\mathrm{d}\eta = \frac{1}{4}(|z|^2-1)^2 - \frac{1}{2}(|z|^2-1), \quad z \in \Omega.$$

Then, some properties for $N_3(z,\zeta)$ hold,

1. $N_3(z,\zeta)$ is triharmonic in $\Omega\backslash\{\zeta\}$ with respect to $z$,
2. $N_3(z,\zeta) + \frac{1}{4}|\zeta - z|^4\log|\zeta - z|^2$ is triharmonic in $\Omega$,
3. $\partial_z\partial_{\overline{z}}N_3(z,\zeta) = N_2(z,\zeta),\ z,\zeta \in \Omega$,
4. $\partial_{\nu_z}N_3(z,\zeta) = \begin{cases} \dfrac{4n}{\pi}\displaystyle\int\limits_\Omega N_2(\zeta,\widetilde{\zeta})\mathrm{d}\widetilde{\xi}\mathrm{d}\widetilde{\eta} = -n(|\zeta|^4 - 4|\zeta|^2+3), & z \in L\backslash\{1,\omega\}, \\ 0, \qquad z \in (0,1)\cup(\omega,0), \end{cases}$
5. $\displaystyle\int\limits_L N_3(z,\zeta)\frac{\mathrm{d}z}{z} = 0$ for $\zeta \in \Omega$.

Therefore, we can transform $N_3(z,\zeta)$ into

$$N_3(z,\zeta) = \frac{1}{4}|\zeta - z|^4 N_1(z,\zeta) + \widehat{h}_3(z,\zeta), \quad z,\zeta \in \Omega, \tag{5.26}$$

where $\widehat{h}_3(z,\zeta)$ is a triharmonic function in $\Omega$. Then, $\widehat{h}_3(z,\zeta)$ satisfies

$$(\partial_z\partial_{\overline{z}})^2\widehat{h}_3(z,\zeta) = g_2(z,\zeta), \quad z,\ \zeta \in \Omega, \tag{5.27}$$

with

$$g_2(z,\zeta) = 2(\overline{\zeta}-\overline{z})\partial_{\overline{z}}N_1 + 2(\zeta - z)\partial_z N_1 - \frac{1}{2}(\overline{\zeta}-\overline{z})^2\partial_{\overline{z}}^2 N_1 - \frac{1}{2}(\zeta - z)^2\partial_z^2 N_1$$
$$= 12n - \sum_{k=0}^{n-1} \mathrm{Re}\Bigg\{ \frac{(\zeta^2-\omega^{-2k})^2}{(\omega^{-2k}-z\zeta)^2} + \frac{(\zeta-\zeta\omega^{2k})^2}{(\zeta - z\omega^{2k})^2} + \frac{(|\zeta|^2-\omega^{-2k})^2}{(\omega^{-2k}-z\overline{\zeta})^2} + \frac{(\overline{\zeta}-\zeta\omega^{2k})^2}{(\overline{\zeta}-z\omega^{2k})^2}$$
$$-2\left[\frac{\zeta^2-\omega^{-2k}}{\omega^{-2k}-z\zeta} + \frac{\zeta\omega^{2k}-\zeta}{\zeta - z\omega^{2k}} + \frac{|\zeta|^2-\omega^{-2k}}{\omega^{-2k}-z\overline{\zeta}} + \frac{\zeta\omega^{2k}-\overline{\zeta}}{\overline{\zeta}-z\omega^{2k}}\right]\Bigg\}, \quad z,\ \zeta \in \Omega, \tag{5.28}$$

and

$$\partial_z\partial_{\overline{z}}\widehat{h}_3(z,\zeta) = -|\zeta - z|^2 N_1 + \frac{1}{2}|\zeta - z|^2\left[(\overline{\zeta}-\overline{z})\partial_{\overline{z}}N_1 + (\zeta - z)\partial_z N_1\right] + N_2, \quad z,\ \zeta\in\Omega. \tag{5.29}$$

Moreover, we have

$$\partial_{\nu_z}\widehat{h}_3(z,\zeta) = \begin{cases} \frac{1}{2}|\zeta - z|^2(z\overline{\zeta}+\overline{z}\zeta - 2)N_1(z,\zeta) \\ \qquad +n[|\zeta-z|^4 - |\zeta|^4 + 4|\zeta|^2 - 3], & z\in L\backslash\{1,\omega\}, \\ -\frac{i}{2}|\zeta - z|^2(\overline{\zeta}-\zeta)N_1(z,\zeta), & z\in(0,1), \\ \frac{i}{2}|\zeta - z|^2(\overline{\zeta}e^{i\theta}-\zeta e^{-i\theta})N_1(z,\zeta), & z\in(\omega,0), \end{cases} \tag{5.30}$$

and

$$\partial_{\nu_z}\partial_z\partial_{\overline{z}}\widehat{h}_3(z,\zeta) = \begin{cases} 4n(|\zeta|^2-1) + (z\overline{\zeta}+\overline{z}\zeta-2)[N_1 - 8n] \\ \quad +\mathrm{Re}\left\{|\zeta-z|^2(z\zeta - z^2)\partial_z^2 N_1 - (\zeta - z)^2\overline{z}\partial_z N_1\right\}, \\ \qquad z\in L\backslash\{1,\omega\}, \\ \mathrm{Im}\left\{2\overline{\zeta}N_1 + |\zeta-z|^2(\zeta-z)\partial_z^2 N_1 + (\zeta-z)^2\partial_z N_1\right\}, \\ \qquad z\in(0,1), \\ \mathrm{Im}\big\{2\zeta e^{-i\theta}N_1 - e^{i\theta}|\zeta-z|^2(\zeta-z)\partial_z^2 N_1 \\ \qquad -e^{-i\theta}(\zeta-z)^2\partial_z N_1\big\}, \quad z\in(\omega,0). \end{cases} \tag{5.31}$$

Then, some particular Neumann problems need to be studied. Firstly define a new triharmonic function

$$
\begin{aligned}
F_2(z,\zeta) &= 3n(|\zeta|^4+|z|^4) - \frac{1}{4}(\overline{z}^2\zeta^2+z^2\overline{\zeta}^2)N_1(z,\zeta) \\
&\quad +\frac{1}{2}(|\zeta|^2+|z|^2)(\overline{z}\zeta+z\overline{\zeta})N_1(z,\zeta) \\
&\quad +\sum_{k=0}^{n-1}\Bigg\{\Bigg[\frac{1}{2}(|z|^2+|\zeta|^2)\Big(\frac{\overline{z}}{\zeta}\omega^{-2k}+\frac{z}{\overline{\zeta}}\omega^{2k}\Big) \\
&\qquad -\frac{1}{4}\Big(\frac{\overline{z}^2}{\zeta^2}\omega^{-4k}+\frac{z^2}{\overline{\zeta}^2}\omega^{4k}\Big)\Bigg]\log|\omega^{-2k}-z\zeta|^2 \\
&\quad +\Bigg[\frac{1}{2}(|z|^2+|\zeta|^2)\Big(\frac{\overline{z}}{\overline{\zeta}}\omega^{-2k}+\frac{z}{\zeta}\omega^{2k}\Big) \\
&\qquad -\frac{1}{4}\Big(\frac{\overline{z}^2}{\overline{\zeta}^2}\omega^{-4k}+\frac{z^2}{\zeta^2}\omega^{4k}\Big)\Bigg]\log|\omega^{-2k}-z\overline{\zeta}|^2 \\
&\quad +\Bigg[\frac{1}{2}(|z|^2+|\zeta|^2)\Big(\overline{z}\zeta\omega^{-2k}+z\overline{\zeta}\omega^{2k}\Big) \\
&\qquad -\frac{1}{4}\Big(\overline{z}^2\zeta^2\omega^{-4k}+z^2\overline{\zeta}^2\omega^{4k}\Big)\Bigg]\log|\zeta-z\omega^{2k}|^2 \\
&\quad +\Bigg[\frac{1}{2}(|z|^2+|\zeta|^2)\Big(\overline{z}\overline{\zeta}\omega^{-2k}+z\zeta\omega^{2k}\Big) \\
&\qquad -\frac{1}{4}\Big(\overline{z}^2\overline{\zeta}^2\omega^{-4k}+z^2\zeta^2\omega^{4k}\Big)\Bigg]\log|\overline{\zeta}-z\omega^{2k}|^2\Bigg\},\ \ z,\ \zeta\in\Omega,
\end{aligned}
\tag{5.32}
$$

and suppose

$$\widetilde{F_2}(z,\zeta)=\widehat{h}_3(z,\zeta)-F_2(z,\zeta),\quad z,\ \zeta\in\Omega. \tag{5.33}$$

Then by computation, we have the following properties for $\widetilde{F_2}(z,\zeta)$,

$$
\begin{cases}
(\partial_z\partial_{\overline{z}})^2\widetilde{F_2}(z,\zeta)=0, & z,\ \zeta\in\Omega,\\
\partial_{\nu_z}\widetilde{F_2}(z,\zeta)=0, & z\in(\omega,0)\cup(0,1),\\
\partial_{\nu_z}\partial_z\partial_{\overline{z}}\widetilde{F_2}(z,\zeta)=0, & z\in(\omega,0)\cup(0,1).
\end{cases}
\tag{5.34}
$$

Furthermore,

$$
\begin{aligned}
\partial_z\partial_{\overline{z}}\widetilde{F_2}(z,\zeta) &= -(|\zeta|^2+1)N_1(z,\zeta)+N_2(z,\zeta)+6n(|\zeta|^2-1) \\
&\quad -\sum_{k=0}^{n-1}\mathrm{Re}\{\Xi_0(z\omega^{2k},\zeta)+\Xi_0(\overline{z}\omega^{-2k},\zeta)\},\quad z\in L,
\end{aligned}
\tag{5.35}
$$

$$\partial_{\nu_z}\widetilde{F_2}(z,\zeta) = -(2|\zeta|^2+1)N_1(z,\zeta) - 14n + 8n|\zeta|^2$$
$$-\sum_{k=0}^{n-1}\mathrm{Re}\{\Xi_1(z\omega^{2k},\zeta)+\Xi_1(\overline{z}\omega^{-2k},\zeta)\}, \quad z\in L\backslash\{1,\omega\}, \tag{5.36}$$

$$\partial_{\nu_z}\partial_z\partial_{\overline{z}}\widetilde{F_2}(z,\zeta) = -2n - \sum_{k=0}^{n-1}\mathrm{Re}\{\Xi_2(z\omega^{2k},\zeta)+\Xi_2(\overline{z}\omega^{-2k},\zeta)\},\ z\in L\backslash\{1,\omega\}, \tag{5.37}$$

where

$$\Xi_0(z,\zeta) \ = \frac{2z(|\zeta|^2+1)}{\zeta}\log|1-z\overline{\zeta}|^2 - \frac{(|\zeta|^2+1)(\overline{z}\zeta^2-z^2\overline{\zeta})}{|\zeta|^2(\zeta-z)}, \tag{5.38}$$

$$\Xi_1(z,\zeta) \ = -z^2\zeta^2 + \frac{z\zeta(|\zeta|^2+1)(|\zeta|^2+3)-z^2(|\zeta|^4+1)}{\zeta^2}\log|1-z\overline{\zeta}|^2$$
$$+2z\zeta(|\zeta|^2+1) + \frac{\zeta}{\zeta-z}\left[\frac{(1-|\zeta|^4)}{|\zeta|^2}(z\overline{\zeta}+\overline{z}\zeta) - \frac{(1-|\zeta|^4)}{2|\zeta|^4}(z^2\overline{\zeta}^2+\overline{z}^2\zeta^2)\right], \tag{5.39}$$

and

$$\Xi_2(z,\zeta) \ = 2\left[\frac{z(|\zeta|^2+1)}{\zeta}-2\right]\log|1-z\overline{\zeta}|^2 - \frac{1}{(\zeta-z)^2}\left[z^3\overline{\zeta} - 2z\zeta|\zeta|^2 + \frac{\overline{z}\zeta^2}{\overline{\zeta}}\right]$$
$$-\frac{1}{\zeta-z}\left[\frac{z^2}{\zeta} + 4z^2\overline{\zeta} - \overline{z}\zeta^2 - \frac{4\overline{z}\zeta}{\overline{\zeta}}\right]. \tag{5.40}$$

Thus, the following theorem is valid.

**Theorem 5.2.1.** *For $z,\zeta\in\Omega$,*

$$\widehat{h}_3(z,\zeta) \ = F_2(z,\zeta) - \frac{3}{2}n(|\zeta|^4+1) + 2n(2|\zeta|^2-1)(2|z|^2-1) + 2n$$
$$+\frac{1}{4\pi i}\int_L \left[N_1(z,\widetilde{\zeta})\partial_{\nu_{\widetilde{\zeta}}}\widetilde{F_2}(\widetilde{\zeta},\zeta) + N_2(z,\widetilde{\zeta})\partial_{\nu_{\widetilde{\zeta}}}\partial_{\widetilde{\zeta}\overline{\widetilde{\zeta}}}\widetilde{F_2}(\widetilde{\zeta},\zeta)\right]\frac{\mathrm{d}\widetilde{\zeta}}{\widetilde{\zeta}},$$

*where $F_2(z,\zeta)$, $\partial_{\nu_{\widetilde{\zeta}}}\widetilde{F_2}(\widetilde{\zeta},\zeta)$, $\partial_{\nu_{\widetilde{\zeta}}}\partial_{\widetilde{\zeta}\overline{\widetilde{\zeta}}}\widetilde{F_2}(\widetilde{\zeta},\zeta)$ are defined in* (5.32), (5.36) *and* (5.37) *respectively.*

**Proof:** First of all, we verify that the first solvability condition for $\widetilde{F_2}(z,\zeta)$ in Theorem 4.3.4 holds, that is,

$$\frac{1}{2\pi}\int_{\partial\Omega}\partial_{\nu_z}\partial_z\partial_{\overline{z}}\widetilde{F_2}(z,\zeta)\mathrm{d}s_z = \frac{2}{\pi}\int_{\Omega}(\partial_z\partial_{\overline{z}})^2\widetilde{F_2}(z,\zeta)\mathrm{d}x\mathrm{d}y, \tag{5.41}$$

with $z = x + iy,\ x,\ y \in \mathbb{R}$. Obviously, by (5.34) and (5.37), the right-hand side in (5.41) vanishes and the left-hand side is

$$\frac{1}{2\pi i}\int\limits_{L} \partial_{\nu_z}\partial_z\partial_{\overline{z}}\widetilde{F_2}(z,\zeta)\frac{\mathrm{d}z}{z} = -1 - \mathrm{Re}\left\{\frac{1}{2\pi i}\int\limits_{|z|=1} \Xi_2(z,\zeta)\frac{\mathrm{d}z}{z}\right\} = 0,$$

which implies that the first condition (5.41) is true.

The second solvability condition is

$$\begin{aligned}&\frac{n}{\pi}\int\limits_{\partial\Omega} \partial_{\nu_z}\widetilde{F_2}(z,\zeta)\mathrm{d}s_z + \frac{n}{\pi}\int\limits_{\partial\Omega}(|z|^2-1)\partial_{\nu_z}\partial_z\partial_{\overline{z}}\widetilde{F_2}(z,\zeta)\mathrm{d}s_z\\ &= 2c_1 + \frac{4n}{\pi}\int\limits_{\Omega}(|z|^2-1)(\partial_z\partial_{\overline{z}})^2\widetilde{F_2}(z,\zeta)\mathrm{d}x\mathrm{d}y,\end{aligned} \tag{5.42}$$

where $c_1$ is obtained from (5.35) and the normalization of $N_1,\ N_2$,

$$c_1 = \frac{n}{\pi i}\int\limits_{L} \partial_z\partial_{\overline{z}}\widetilde{F_2}(z,\zeta)\frac{\mathrm{d}z}{z} = 6n(|\zeta|^2-1) - \mathrm{Re}\left\{\frac{n}{\pi i}\int\limits_{|z|=1} \Xi_0(z,\zeta)\frac{\mathrm{d}z}{z}\right\} = 4n(2|\zeta|^2-1).$$

In addition, from (5.34), (5.36) and (5.39),

$$\frac{n}{\pi}\int\limits_{\partial\Omega} \partial_{\nu_z}\widetilde{F_2}(z,\zeta)\mathrm{d}s_z = 8n|\zeta|^2 - 14n - \mathrm{Re}\left\{\frac{n}{\pi i}\int\limits_{|z|=1} \Xi_1(z,\zeta)\frac{\mathrm{d}z}{z}\right\} = 8n(2|\zeta|^2-1).$$

Thus, the second solvability condition (5.42) is valid. We also need to calculate

$$\begin{aligned}c_0 &= \frac{n}{\pi i}\int\limits_{L} \widetilde{F_2}(z,\zeta)\frac{\mathrm{d}z}{z} = -3n(|\zeta|^4+1)\\ &\quad -\mathrm{Re}\left\{\frac{n}{\pi i}\int\limits_{|z|=1}\left[\frac{z(1+|\zeta|^2)^2}{\zeta} - \frac{z^2(|\zeta|^4+1)}{2\zeta^2}\right]\log|\zeta - z|^2\frac{\mathrm{d}z}{z}\right\}\\ &= -\frac{3}{2}n(|\zeta|^4+1) + 4n|\zeta|^2.\end{aligned}$$

Hence, from (5.34) and Theorem 4.3.4,

$$\begin{aligned}\widetilde{F_2}(z,\zeta) &= -\frac{3}{2}n(|\zeta|^4+1) + 2n(2|\zeta|^2-1)(2|z|^2-1) + 2n\\ &\quad + \frac{1}{4\pi i}\int\limits_{L}\left[N_1(z,\widetilde{\zeta})\partial_{\nu_{\widetilde{\zeta}}}\widetilde{F_2}(\widetilde{\zeta},\zeta) + N_2(z,\widetilde{\zeta})\partial_{\nu_{\widetilde{\zeta}}}\partial_{\widetilde{\zeta}\overline{\widetilde{\zeta}}}\widetilde{F_2}(\widetilde{\zeta},\zeta)\right]\frac{\mathrm{d}\widetilde{\zeta}}{\widetilde{\zeta}}.\end{aligned}$$

That is, the desired conclusion is true.

Next, we need the following two results.

**Lemma 5.2.1.** *For $z, \zeta \in \Omega$,*

$$
\begin{aligned}
&K_3(z,\zeta) = \frac{1}{4\pi i}\int\limits_L N_1(z,\widetilde{\zeta})\partial_{\nu_{\widetilde{\zeta}}}\widetilde{F_2}(\widetilde{\zeta},\zeta)\frac{\mathrm{d}\widetilde{\zeta}}{\widetilde{\zeta}} \\
&= 2n(|\zeta|^2+1)(|\zeta|^2+3) - \frac{1}{2}n(|\zeta|^4+1) \\
&+ \left[\frac{1-|\zeta|^4}{4|\zeta|^2} - \frac{1+|\zeta|^4}{4|z|^2|\zeta|^2} + \frac{1}{2}\right](z+\overline{z})(\zeta+\overline{\zeta})B_0 - 2(2|\zeta|^2+1) \\
&\times \sum_{k=0}^{n-1}\sum_{l=0}^{\infty}\frac{1}{(l+1)^2}\left[(z\zeta\omega^{2k})^{l+1} + (\overline{z}\zeta\omega^{-2k})^{l+1} + (z\overline{\zeta}\omega^{2k})^{l+1} + (\overline{z}\overline{\zeta}\omega^{-2k})^{l+1}\right] \\
&+ \sum_{k=0}^{n-1}\left\{\frac{1-|\zeta|^4}{|\zeta|^2}\log|(1-z\overline{\zeta}\omega^{2k})(1-\overline{z}\overline{\zeta}\omega^{-2k})|^2 + \frac{1}{2}(|\zeta|^2+1)(|\zeta|^2+3)\right. \\
&\times \left[\left(\frac{1}{\overline{z}\zeta\omega^{-2k}} - \frac{\overline{z}\omega^{-2k}}{\overline{\zeta}}\right)\log(1-\overline{z}\zeta\omega^{-2k}) + \left(\frac{1}{z\overline{\zeta}\omega^{2k}} - \frac{z\omega^{2k}}{\zeta}\right)\log(1-z\overline{\zeta}\omega^{2k})\right. \\
&+ \left.\left(\frac{1}{z\zeta\omega^{2k}} - \frac{z\omega^{2k}}{\overline{\zeta}}\right)\log(1-z\zeta\omega^{2k}) + \left(\frac{1}{\overline{z}\overline{\zeta}\omega^{-2k}} - \frac{\overline{z}\omega^{-2k}}{\zeta}\right)\log(1-\overline{z}\overline{\zeta}\omega^{-2k})\right] \\
&+ \frac{1}{4}(1+|\zeta|^4)\left[\left(\frac{\overline{z}^2\omega^{-4k}}{\overline{\zeta}^2} - \frac{\omega^{4k}}{\overline{z}^2\zeta^2}\right)\log(1-\overline{z}\zeta\omega^{-2k})\right. \\
&+ \left(\frac{z^2\omega^{4k}}{\zeta^2} - \frac{\omega^{-4k}}{z^2\overline{\zeta}^2}\right)\log(1-z\overline{\zeta}\omega^{2k}) + \left(\frac{z^2\omega^{4k}}{\overline{\zeta}^2} - \frac{\omega^{-4k}}{z^2\zeta^2}\right)\log(1-z\zeta\omega^{2k}) \\
&\qquad\left.\left.+ \left(\frac{\overline{z}^2\omega^{-4k}}{\zeta^2} - \frac{\omega^{4k}}{\overline{z}^2\overline{\zeta}^2}\right)\log(1-\overline{z}\overline{\zeta}\omega^{-2k})\right]\right\},
\end{aligned}
$$

*where $B_0$ is given by* (4.33).

**Proof:** From (5.36), the above expression follows from

$$
\begin{aligned}
&\frac{1}{4\pi i}\int\limits_L N_1(z,\widetilde{\zeta})\partial_{\nu_{\widetilde{\zeta}}}\widetilde{F_2}(\widetilde{\zeta},\zeta)\frac{\mathrm{d}\widetilde{\zeta}}{\widetilde{\zeta}} \\
&= \frac{1}{4\pi i}\int\limits_L N_1(z,\widetilde{\zeta})\left[8n|\zeta|^2 - 14n - (2|\zeta|^2+1)N_1(\widetilde{\zeta},\zeta)\right]\frac{\mathrm{d}\widetilde{\zeta}}{\widetilde{\zeta}} \\
&\qquad -\mathrm{Re}\left\{\frac{1}{4\pi i}\int\limits_{|\widetilde{\zeta}|=1} N_1(z,\widetilde{\zeta})\Xi_1(\widetilde{\zeta},\zeta)\frac{\mathrm{d}\widetilde{\zeta}}{\widetilde{\zeta}}\right\},
\end{aligned}
$$

where $\Xi_1(z,\zeta)$ is given by (5.39).

**Lemma 5.2.2.** *For* $z,\zeta\in\Omega$,

$$
\begin{aligned}
K_4(z,\zeta) &= \frac{1}{4\pi i}\int\limits_L N_2(z,\widetilde{\zeta})\partial_{\nu_{\widetilde{\zeta}}}\partial_{\widetilde{\zeta}\overline{\widetilde{\zeta}}}\widetilde{F_2}(\widetilde{\zeta},\zeta)\frac{d\widetilde{\zeta}}{\widetilde{\zeta}}\\
&= 2n(|\zeta|^2+1)(|z|^2+1)+8n(|\zeta|^2+2|z|^2+3)\\
&\qquad -\frac{(1+|\zeta|^2)(1+|z|^2)}{4|\zeta|^2}(z+\overline{z})(\zeta+\overline{\zeta})B_0\\
&\quad +\sum_{k=0}^{n-1}\sum_{l=0}^{\infty}\left[\frac{8}{(l+1)^3}-\frac{4(1+|z|^2)}{(l+1)^2}-\frac{(|z|^2+1)(|\zeta|^2+1)}{(l+2)^2}\right]\\
&\quad \times\left[(z\zeta\omega^{2k})^{l+1}+(\overline{z}\zeta\omega^{-2k})^{l+1}+(z\overline{\zeta}\omega^{2k})^{l+1}+(\overline{z}\overline{\zeta}\omega^{-2k})^{l+1}\right]\\
&\quad +\sum_{k=0}^{n-1}\left\{\left[\frac{1}{2}(|\zeta|^2+1)(|z|^2+1)+2(2|z|^2+|\zeta|^2+3)\right]\right.\\
&\quad \times\left[\frac{\log(1-\overline{z}\zeta\omega^{-2k})}{\overline{z}\zeta\omega^{-2k}}+\frac{\log(1-z\overline{\zeta}\omega^{2k})}{z\overline{\zeta}\omega^{2k}}+\frac{\log(1-z\zeta\omega^{2k})}{z\zeta\omega^{2k}}+\frac{\log(1-\overline{z}\overline{\zeta}\omega^{-2k})}{\overline{z}\overline{\zeta}\omega^{-2k}}\right]\\
&\quad +\frac{(1+|\zeta|^2)(3-|z|^2)}{2|\zeta|^2}\left[\overline{z}\zeta\omega^{-2k}\log(1-\overline{z}\zeta\omega^{-2k})+z\overline{\zeta}\omega^{2k}\log(1-z\overline{\zeta}\omega^{2k})\right.\\
&\quad \left. +z\zeta\omega^{2k}\log(1-z\zeta\omega^{2k})+\overline{z}\overline{\zeta}\omega^{-2k}\log(1-\overline{z}\overline{\zeta}\omega^{-2k})\right]\\
&\quad \left. -\left[\frac{(1+|\zeta|^2)^2}{|\zeta|^2}+2(2|z|^2+|\zeta|^2+3)\right]\log|(1-z\overline{\zeta}\omega^{2k})(1-z\zeta\omega^{2k})|^2\right\},
\end{aligned}
$$

*with* $B_0$ *given in* (4.33).

**Proof:** Similarly by (5.37), the result derives from

$$
\begin{aligned}
&\frac{1}{4\pi i}\int\limits_L N_2(z,\widetilde{\zeta})\partial_{\nu_{\widetilde{\zeta}}}\partial_{\widetilde{\zeta}\overline{\widetilde{\zeta}}}\widetilde{F_2}(\widetilde{\zeta},\zeta)\frac{d\widetilde{\zeta}}{\widetilde{\zeta}}\\
&= -\frac{n}{2\pi i}\int\limits_L N_2(z,\widetilde{\zeta})\frac{d\widetilde{\zeta}}{\widetilde{\zeta}}-\mathrm{Re}\left\{\frac{1}{4\pi i}\int\limits_{|\widetilde{\zeta}|=1}N_2(z,\widetilde{\zeta})\Xi_2(\widetilde{\zeta},\zeta)\frac{d\widetilde{\zeta}}{\widetilde{\zeta}}\right\},
\end{aligned}
$$

where $\Xi_2(z,\zeta)$ is given by (5.40).

Therefore, by Theorem 5.2.1,

$$\begin{aligned}\widehat{h}_3(z,\zeta) &= -\frac{3}{2}n(|\zeta|^4+1)+2n(2|\zeta|^2-1)(2|z|^2-1) \\ &\quad +2n+K_3(z,\zeta)+K_4(z,\zeta)+F_2(z,\zeta),\end{aligned} \tag{5.43}$$

with $K_3(z,\zeta)$, $K_4(z,\zeta)$ determined by Lemmas 5.2.1-5.2.2 respectively. Hence, also from (5.26),

$$\begin{aligned}N_3(z,\zeta) &= 3n(|\zeta|^4+|z|^4)+10n(|\zeta|^2+1)(|z|^2+1)+4n(|\zeta|^2+|z|^2+6) \\ &\quad -\frac{1}{4}\left[\frac{|\zeta|^4+|z|^4+1}{|z|^2|\zeta|^2}+|\zeta|^2+|z|^2-1\right](z+\overline{z})(\zeta+\overline{\zeta})B_0 \\ &\quad +\sum_{k=0}^{n-1}\sum_{l=0}^{\infty}\Bigg\{\left[\frac{8}{(l+1)^3}-\frac{(|\zeta|^2+1)(|z|^2+1)}{(l+2)^2}-\frac{4|z|^2+4|\zeta|^2+6}{(l+1)^2}\right] \\ &\quad \times\left[(z\zeta\omega^{2k})^{l+1}+(\overline{z}\zeta\omega^{-2k})^{l+1}+(z\overline{\zeta}\omega^{2k})^{l+1}+(\overline{z}\overline{\zeta}\omega^{-2k})^{l+1}\right]\Bigg\} \\ &\quad +\sum_{k=0}^{n-1}\Bigg\{\frac{1}{4}|\zeta-z\omega^{2k}|^4\log\left|\frac{1-z\overline{\zeta}\omega^{2k}}{\zeta-z\omega^{2k}}\right|^2+\frac{1}{4}|\zeta-\overline{z}\omega^{-2k}|^4\log\left|\frac{1-z\zeta\omega^{2k}}{\zeta-\overline{z}\omega^{-2k}}\right|^2 \\ &\quad -\left[2(|z|^2+2)(|\zeta|^2+2)+\frac{1}{2}(|\zeta|^4+|z|^4)\right]\log|(1-z\zeta\omega^{2k})(1-z\overline{\zeta}\omega^{2k})|^2 \\ &\quad +\left[\frac{1}{2}(|\zeta|^2+|z|^2)(|z|^2+1)(|\zeta|^2+1)+4(|z|^2+|\zeta|^2+2)\right] \\ &\quad \times\Bigg[\frac{\log(1-\overline{z}\zeta\omega^{-2k})}{\overline{z}\zeta\omega^{-2k}}+\frac{\log(1-z\overline{\zeta}\omega^{2k})}{z\overline{\zeta}\omega^{2k}} \\ &\qquad +\frac{\log(1-z\zeta\omega^{2k})}{z\zeta\omega^{2k}}+\frac{\log(1-\overline{z}\overline{\zeta}\omega^{-2k})}{\overline{z}\overline{\zeta}\omega^{-2k}}\Bigg] \\ &\quad -\frac{1}{4}(|z|^4+1)(|\zeta|^4+1)\Bigg[\frac{\log(1-\overline{z}\zeta\omega^{-2k})}{(\overline{z}\zeta\omega^{-2k})^2}+\frac{\log(1-z\overline{\zeta}\omega^{2k})}{(z\overline{\zeta}\omega^{2k})^2} \\ &\qquad +\frac{\log(1-z\zeta\omega^{2k})}{(z\zeta\omega^{2k})^2}+\frac{\log(1-\overline{z}\overline{\zeta}\omega^{-2k})}{(\overline{z}\overline{\zeta}\omega^{-2k})^2}\Bigg]\Bigg\},\end{aligned} \tag{5.44}$$

where $B_0$ is given in (4.33). By computation, the expression (5.44) satisfies

$$(\partial_z\partial_{\overline{z}})^2N_3(z,\zeta)=N_1(z,\zeta),\quad z,\ \zeta\in\Omega,$$

$$\partial_{\nu_z} N_3(z,\zeta) = \begin{cases} -n(|\zeta|^4 - 4|\zeta|^2 + 3), & z \in L\backslash\{1,\omega\}, \\ 0, & z \in (0,1)\cup(\omega,0), \end{cases} \quad \text{for } \ \zeta \in \Omega,$$

$$\partial_{\nu_z}\partial_z\partial_{\overline{z}} N_3(z,\zeta) = \begin{cases} -4n(|\zeta|^2 - 1), & z \in L\backslash\{1,\omega\}, \\ 0, & z \in (0,1)\cup(\omega,0), \end{cases} \quad \text{for } \ \zeta \in \Omega.$$

Also, the normalization condition is valid,

$$\frac{1}{2\pi i}\int_L N_3(z,\zeta)\frac{\mathrm{d}z}{z} = \frac{3}{2}n(|\zeta|^4+1) + 12n(|\zeta|^2+2)$$
$$+\mathrm{Re}\left\{\frac{2\left[(|\zeta|^2+1)^2+4(|\zeta|^2+3)\right]}{2\pi i}\int_{|z|=1}\frac{\log(1-z\zeta)}{z\zeta}\frac{\mathrm{d}z}{z}\right.$$
$$\left. -\frac{(|\zeta|^4+1)}{2\pi i}\int_{|z|=1}\frac{\log(1-z\zeta)}{z^2\zeta^2}\frac{\mathrm{d}z}{z}\right\} = 0.$$

Therefore, the expression (5.44) is exactly the desired triharmonic Neumann function for the domain $\Omega$.

## 5.3 Triharmonic Boundary Value Problems

Obviously, as in [27], we obtain the following representation formula and the solution to the Dirichlet problem.

**Theorem 5.3.1.** *Any $w \in C^6(\Omega;\mathbb{C}) \cap C^5(\overline{\Omega};\mathbb{C})$ can be expressed as*

$$w(z) = -\sum_{\mu=1}^{3}\frac{1}{4\pi}\int_{\partial\Omega}\partial_{\nu_\zeta}\widehat{G}_\mu(z,\zeta)(\partial_\zeta\partial_{\overline{\zeta}})^{\mu-1}w(\zeta)\mathrm{d}s_\zeta - \frac{1}{\pi}\int_\Omega \widehat{G}_3(z,\zeta)(\partial_\zeta\partial_{\overline{\zeta}})^3 w(\zeta)\mathrm{d}\xi\mathrm{d}\eta,$$

*where $\widehat{G}_1(z,\zeta) = G_1(z,\zeta)$ and $\widehat{G}_\mu(z,\zeta)$ $(\mu = 2,3)$ are the harmonic Green, the biharmonic Green and the triharmonic Green functions for $\Omega$ respectively.*

**Theorem 5.3.2.** *The Dirichlet problem*

$$\begin{cases} (\partial_z\partial_{\overline{z}})^3 w(z) = f \quad in \ \ \Omega, \\ w = \gamma_0, \quad \partial_z\partial_{\overline{z}} w = \gamma_1, \quad (\partial_z\partial_{\overline{z}})^2 w = \gamma_2 \quad on \ \ \partial\Omega, \end{cases}$$

*with* $f \in L_p(\Omega;\mathbb{C}),\ p>2,\ \gamma_0,\ \gamma_1,\ \gamma_2 \in C(\partial\Omega;\mathbb{C})$ *is uniquely solvable by*

$$w(z) = -\frac{1}{4\pi}\int\limits_{\partial\Omega}\Big[\partial_{\nu_\zeta}G_1(z,\zeta)\gamma_0(\zeta)+\partial_{\nu_\zeta}\widehat{G}_2(z,\zeta)\gamma_1(\zeta)+\partial_{\nu_\zeta}\widehat{G}_3(z,\zeta)\gamma_2(\zeta)\Big]\mathrm{d}s_\zeta$$
$$-\frac{1}{\pi}\int\limits_{\Omega}\widehat{G}_3(z,\zeta)f(\zeta)\mathrm{d}\xi\mathrm{d}\eta,\quad z\in\Omega,$$

*where* $G_1(z,\zeta),\ \widehat{G}_2(z,\zeta),\ \widehat{G}_3(z,\zeta)$ *are given by* (4.1), (4.16) *and* (5.21) *respectively.*

Next, we give another representation formula and the statement for the Neumann problem related to $N_i(z,\zeta)$ $(i=1,2,3)$.

**Theorem 5.3.3.** *Any* $w\in C^6(\Omega;\mathbb{C})\cap C^5(\overline{\Omega};\mathbb{C})$ *can be represented as*

$$\begin{aligned} w(z) &= \frac{n}{\pi i}\int\limits_L w(\zeta)\frac{\mathrm{d}\zeta}{\zeta}+\frac{n(|z|^2-1)}{\pi i}\int\limits_L \partial_\zeta\partial_{\overline{\zeta}}w(\zeta)\frac{\mathrm{d}\zeta}{\zeta}\\ &+\left[\frac{1}{4}(|z|^2-1)^2-\frac{1}{2}(|z|^2-1)\right]\frac{n}{\pi i}\int\limits_L(\partial_\zeta\partial_{\overline{\zeta}})^2w(\zeta)\frac{\mathrm{d}\zeta}{\zeta}\\ &+\frac{1}{4\pi}\int\limits_{\partial\Omega}\big[N_1(z,\zeta)\partial_{\nu_\zeta}w(\zeta)+N_2(z,\zeta)\partial_{\nu_\zeta}(\partial_\zeta\partial_{\overline{\zeta}})w(\zeta)\\ &\qquad+N_3(z,\zeta)\partial_{\nu_\zeta}(\partial_\zeta\partial_{\overline{\zeta}})^2w(\zeta)\big]\mathrm{d}s_\zeta-\frac{1}{\pi}\int\limits_\Omega N_3(z,\zeta)(\partial_\zeta\partial_{\overline{\zeta}})^3w(\zeta)\mathrm{d}\xi\mathrm{d}\eta,\end{aligned}$$

*where* $N_i(z,\zeta)$ $(i=1,2,3)$ *are the harmonic Nuemann, the biharmonic Neumann and the triharmonic Neumann functions respectively.*

**Proof:** Applying the representation formula in Theorem 4.3.3 to $\partial_z\partial_{\overline{z}}w(z)$ gives

$$\begin{aligned}\partial_z\partial_{\overline{z}}w(z) &= \frac{n}{\pi i}\int\limits_L\partial_\zeta\partial_{\overline{\zeta}}w(\zeta)\frac{\mathrm{d}\zeta}{\zeta}+\frac{n(|z|^2-1)}{\pi i}\int\limits_L(\partial_\zeta\partial_{\overline{\zeta}})^2w(\zeta)\frac{\mathrm{d}\zeta}{\zeta}\\ &+\frac{1}{4\pi}\int\limits_{\partial\Omega}\left[N_1(z,\zeta)\partial_{\nu_\zeta}\partial_\zeta\partial_{\overline{\zeta}}w(\zeta)+N_2(z,\zeta)\partial_{\nu_\zeta}(\partial_\zeta\partial_{\overline{\zeta}})^2w(\zeta)\right]\mathrm{d}s_\zeta\\ &-\frac{1}{\pi}\int\limits_\Omega N_2(z,\zeta)(\partial_\zeta\partial_{\overline{\zeta}})^3w(\zeta)\mathrm{d}\xi\mathrm{d}\eta.\end{aligned} \tag{5.45}$$

Then, we substitute (5.45) into the following representation formula

$$w(z) = \frac{1}{4\pi}\int\limits_{\partial\Omega}\left[\partial_{\nu_\zeta} w(\zeta)N_1(z,\zeta) - w(\zeta)\partial_{\nu_\zeta}N_1(z,\zeta)\right]\mathrm{d}s_\zeta \\ -\frac{1}{\pi}\int\limits_{\Omega} N_1(z,\zeta)\partial_\zeta\partial_{\overline{\zeta}}w(\zeta)\mathrm{d}\xi\mathrm{d}\eta, \tag{5.46}$$

thus, the result follows from

$$\begin{cases} -\dfrac{1}{\pi}\displaystyle\int\limits_{\Omega} N_1(z,\zeta)\mathrm{d}\xi\mathrm{d}\eta = |z|^2 - 1, \\ -\dfrac{1}{\pi}\displaystyle\int\limits_{\Omega}(|\zeta|^2-1)N_1(z,\zeta)\mathrm{d}\xi\mathrm{d}\eta = \dfrac{1}{4}(|z|^2-1)^2 - \dfrac{1}{2}(|z|^2-1). \end{cases} \tag{5.47}$$

**Theorem 5.3.4.** *The Neumann problem*

$$(\partial_z\partial_{\overline{z}})^3 w = f \quad in \quad \Omega, \quad \partial_{\nu_z} w = \gamma_0, \quad \partial_{\nu_z}\partial_z\partial_{\overline{z}}w = \gamma_1, \quad \partial_{\nu_z}(\partial_z\partial_{\overline{z}})^2 w = \gamma_2 \ on \ \partial\Omega,$$

$$\frac{n}{\pi i}\int\limits_L w(\zeta)\frac{\mathrm{d}\zeta}{\zeta} = c_0, \quad \frac{n}{\pi i}\int\limits_L \partial_\zeta\partial_{\overline{\zeta}}w(\zeta)\frac{\mathrm{d}\zeta}{\zeta} = c_1, \quad \frac{n}{\pi i}\int\limits_L (\partial_\zeta\partial_{\overline{\zeta}})^2 w(\zeta)\frac{\mathrm{d}\zeta}{\zeta} = c_2$$

*for* $f \in L_p(\Omega;\mathbb{C}),\ p > 2,\ \gamma_1,\ \gamma_2,\ \gamma_3 \in C(\partial\Omega;\mathbb{C})$, *is solvable if and only if*

$$\frac{1}{2\pi}\int\limits_{\partial\Omega}\gamma_2(\zeta)\mathrm{d}s_\zeta = \frac{2}{\pi}\int\limits_{\Omega} f(\zeta)\mathrm{d}\xi\mathrm{d}\eta, \tag{5.48}$$

$$\frac{n}{\pi}\int\limits_{\partial\Omega}\gamma_1(\zeta)\mathrm{d}s_\zeta + \frac{n}{\pi}\int\limits_{\partial\Omega}(|\zeta|^2-1)\gamma_2(\zeta)\mathrm{d}s_\zeta = 2c_2 + \frac{4n}{\pi}\int\limits_{\Omega}(|\zeta|^2-1)f(\zeta)\mathrm{d}\xi\mathrm{d}\eta, \tag{5.49}$$

*and*

$$\frac{n}{\pi}\int\limits_{\partial\Omega}\gamma_0(\zeta)\mathrm{d}s_\zeta + \frac{n}{\pi}\int\limits_{\partial\Omega}(|\zeta|^2-1)\left[\gamma_1(\zeta) + \frac{1}{4}\gamma_2(\zeta)(|\zeta|^2-3)\right]\mathrm{d}s_\zeta \\ = 2c_1 - c_2 + \frac{n}{\pi}\int\limits_{\Omega}(|\zeta|^2-1)(|\zeta|^2-3)f(\zeta)\mathrm{d}\xi\mathrm{d}\eta. \tag{5.50}$$

*Then its solution is uniquely expressed as*

$$\begin{aligned} w(z) &= c_0 + (|z|^2-1)c_1 + \left[\frac{1}{4}(|z|^2-1)^2 - \frac{1}{2}(|z|^2-1)\right] c_2 \\ &+\frac{1}{4\pi}\int\limits_{\partial\Omega} [N_1(z,\zeta)\gamma_0(\zeta) + N_2(z,\zeta)\gamma_1(\zeta) + N_3(z,\zeta)\gamma_2(\zeta)]\, \mathrm{d}s_\zeta \\ &-\frac{1}{\pi}\int\limits_{\Omega} N_3(z,\zeta)f(\zeta)\mathrm{d}\xi\mathrm{d}\eta, \quad z\in\Omega. \end{aligned} \tag{5.51}$$

*where* $N_1(z,\zeta),\ N_2(z,\zeta),\ N_3(z,\zeta)$ *are given in* (4.17), (4.37) *and* (5.44) *respectively.*

**Proof:** Rewriting the above Neumann problem as the following system

$$(\partial_z\partial_{\overline{z}})^2 w = w_1 \ \text{ in } \ \Omega, \ \ \partial_{\nu_z} w = \gamma_0, \ \ \partial_{\nu_z}\partial_z\partial_{\overline{z}} w = \gamma_1 \ \text{ on } \ \partial\Omega,$$

$$\frac{n}{\pi i}\int\limits_L w(\zeta)\frac{\mathrm{d}\zeta}{\zeta} = c_0, \ \ \frac{n}{\pi i}\int\limits_L \partial_\zeta\partial_{\overline{\zeta}} w(\zeta)\frac{\mathrm{d}\zeta}{\zeta} = c_1,$$

and

$$\partial_z\partial_{\overline{z}} w_1 = f \ \text{ in } \ \Omega, \ \ \partial_{\nu_z} w_1 = \gamma_2, \ \ \frac{n}{\pi i}\int\limits_L w_1(\zeta)\frac{\mathrm{d}\zeta}{\zeta} = c_2.$$

Hence, from Theorem 4.3.4 and Lemma 4.2.1, if and only if

$$\frac{1}{2\pi}\int\limits_{\partial\Omega}\gamma_2(\zeta)\mathrm{d}s_\zeta = \frac{2}{\pi}\int\limits_{\Omega} f(\zeta)\mathrm{d}\xi\mathrm{d}\eta, \tag{5.52}$$

$$\frac{1}{2\pi}\int\limits_{\partial\Omega}\gamma_1(\zeta)\mathrm{d}s_\zeta = \frac{2}{\pi}\int\limits_{\Omega} w_1(\zeta)\mathrm{d}\xi\mathrm{d}\eta \tag{5.53}$$

and

$$\frac{n}{\pi}\int\limits_{\partial\Omega}\gamma_0(\zeta)\mathrm{d}s_\zeta + \frac{n}{\pi}\int\limits_{\partial\Omega}(|\zeta|^2-1)\gamma_1(\zeta)\mathrm{d}s_\zeta = 2c_1 + \frac{4n}{\pi}\int\limits_{\Omega}(|\zeta|^2-1)w_1(\zeta)\mathrm{d}\xi\mathrm{d}\eta \tag{5.54}$$

are satisfied, then $w,\ w_1$ are uniquely expressed as, respectively,

$$w_1(z) = c_2 + \frac{1}{4\pi}\int\limits_{\partial\Omega} N_1(z,\zeta)\gamma_2(\zeta)\mathrm{d}s_\zeta - \frac{1}{\pi}\int\limits_{\Omega} N_1(z,\zeta)f(\zeta)\mathrm{d}\xi\mathrm{d}\eta \tag{5.55}$$

and

$$
\begin{aligned}
w(z) = c_0 + (|z|^2 - 1)c_1 + \frac{1}{4\pi} \int\limits_{\partial\Omega} [N_1(z,\zeta)\gamma_0(\zeta) + N_2(z,\zeta)\gamma_1(\zeta)]\, \mathrm{d}s_\zeta \\
-\frac{1}{\pi} \int\limits_{\Omega} N_2(z,\zeta) w_1(\zeta) \mathrm{d}\xi \mathrm{d}\eta.
\end{aligned}
\tag{5.56}
$$

Thus, putting (5.55) into (5.53) and (5.54) respectively, the desired solvability conditions (5.49) and (5.50) follow from (5.47) and

$$
-\frac{1}{\pi} \int\limits_{\Omega} (|\zeta|^2 - 1) \mathrm{d}\xi \mathrm{d}\eta = \frac{1}{4n}.
$$

Similarly, putting (5.55) into (5.56) gives that the unique solution (5.51) is also valid from

$$
-\frac{1}{\pi} \int\limits_{\Omega} N_2(z,\zeta) \mathrm{d}\xi \mathrm{d}\eta = \frac{1}{4}(|z|^2 - 1)^2 - \frac{1}{2}(|z|^2 - 1).
$$

Then the proof is completed.

# Chapter 6

# Tetra-harmonic Boundary Value Problems

On the basis of m-harmonic Green and Neumann functions ($m = 1, 2, 3$), we seek to find a tetra-harmonic Green function and a tetra-harmonic Neumann function for the domain $\Omega$, and then study the related tetra-harmonic boundary value problems. Also, $\Omega,\ L,\ \omega, \theta$ are defined as before.

## 6.1 Tetra-harmonic Dirichlet Problem

A tetra-harmonic Green function $\widehat{G}_4(z,\zeta)$ determined later for the domain $\Omega$ should satisfy the properties,

1. $\widehat{G}_4(z,\zeta)$ is tetra-harmonic in $\Omega\backslash\{\zeta\}$,

2. $\widehat{G}_4(z,\zeta) + \dfrac{1}{36}|\zeta - z|^6 \log|\zeta - z|^2$ is tetra-harmonic in $\Omega$,

3. $(\partial_z\partial_{\bar{z}})^3\widehat{G}_4(z,\zeta) = G_1(z,\zeta)$ for $z \in \Omega$,

$(\partial_z\partial_{\bar{z}})^{j-1}\widehat{G}_4(z,\zeta) = 0, \quad j = 1, 2, 3, 4,\ z \in \partial\Omega$,

4. $\widehat{G}_4(z,\zeta) = \widehat{G}_4(\zeta, z)$ for $z \neq \zeta$.

Actually, $\widehat{G}_4(z,\zeta)$ is equivalent to another expression,

$$\widehat{G}_4(z,\zeta) = -\frac{1}{\pi}\int\limits_{\Omega} G_1(z,\widetilde{\zeta})\widehat{G}_3(\widetilde{\zeta},\zeta)d\widetilde{\xi}d\widetilde{\eta}, \quad z,\ \zeta \in \Omega, \tag{6.1}$$

where $G_1,\ \widehat{G}_3$ are defined in (4.1) and (5.21) respectively. We represent $\widehat{G}_4(z,\zeta)$ as

$$\widehat{G}_4(z,\zeta) = \frac{1}{36}|\zeta - z|^6 G_1(z,\zeta) + h_4(z,\zeta), \quad z,\zeta \in \Omega, \tag{6.2}$$

with $h_4(z,\zeta)$ being a tetra-harmonic function. Then, $h_4(z,\zeta)$ satisfies the properties,

$$(\partial_z\partial_{\bar{z}})^3 h_4(z,\zeta) = f_2(z,\zeta), \quad z \in \Omega, \qquad h_4(z,\zeta) = 0, \quad z \in \partial\Omega, \tag{6.3}$$

and for $z \in \partial\Omega$,

$$\partial_z \partial_{\overline{z}} h_4(z,\zeta) = \frac{1}{12}|\zeta - z|^4 \left[ (\overline{\zeta} - \overline{z}) \partial_{\overline{z}} G_1(z,\zeta) + (\zeta - z) \partial_z G_1(z,\zeta) \right], \tag{6.4}$$

$$\begin{aligned} (\partial_z \partial_{\overline{z}})^2 h_4(z,\zeta) &= |\zeta - z|^2 \left[ (\zeta - z) \partial_z G_1(z,\zeta) + (\overline{\zeta} - \overline{z}) \partial_{\overline{z}} G_1(z,\zeta) \right] \\ &\quad - \frac{1}{6}|\zeta - z|^2 \left[ (\overline{\zeta} - \overline{z})^2 \partial_{\overline{z}}^2 G_1(z,\zeta) + (\zeta - z)^2 \partial_z^2 G_1(z,\zeta) \right], \end{aligned} \tag{6.5}$$

where precisely,

$$\begin{aligned} f_2(z,\zeta) &= \sum_{k=0}^{n-1} \mathrm{Re} \Bigg\{ -2 \left[ \frac{|\zeta|^2 - \omega^{-2k}}{\omega^{-2k} - z\overline{\zeta}} + \frac{\zeta\omega^{2k} - \overline{\zeta}}{\overline{\zeta} - z\omega^{2k}} - \frac{\zeta^2 - \omega^{-2k}}{\omega^{-2k} - z\zeta} - \frac{\zeta\omega^{2k} - \zeta}{\zeta - z\omega^{2k}} \right] \\ &\quad - \frac{2}{3} \left[ \frac{(|\zeta|^2 - \omega^{-2k})^3}{(\omega^{-2k} - z\overline{\zeta})^3} + \frac{(\zeta\omega^{2k} - \overline{\zeta})^3}{(\overline{\zeta} - z\omega^{2k})^3} - \frac{(\zeta^2 - \omega^{-2k})^3}{(\omega^{-2k} - z\zeta)^3} - \frac{(\zeta\omega^{2k} - \zeta)^3}{(\zeta - z\omega^{2k})^3} \right] \\ &\quad + \frac{(|\zeta|^2 - \omega^{-2k})^2}{(\omega^{-2k} - z\overline{\zeta})^2} + \frac{(\zeta\omega^{2k} - \overline{\zeta})^2}{(\overline{\zeta} - z\omega^{2k})^2} - \frac{(\zeta^2 - \omega^{-2k})^2}{(\omega^{-2k} - z\zeta)^2} - \frac{(\zeta\omega^{2k} - \zeta)^2}{(\zeta - z\omega^{2k})^2} \Bigg\}. \end{aligned} \tag{6.6}$$

Introducing a tetra-harmonic function

$$\begin{aligned} F_4(z,\zeta) &= \frac{1}{36}(\overline{z}\zeta + z\overline{\zeta})^3 G_1(z,\zeta) + \frac{1}{12}(\overline{z}\zeta + z\overline{\zeta})(|\zeta|^2 + |z|^2)^2 G_1(z,\zeta) \\ &\quad - \frac{1}{12}(\overline{z}^2\zeta^2 + z^2\overline{\zeta}^2)(|z|^2 + |\zeta|^2) G_1(z,\zeta) \\ &\quad + \sum_{k=0}^{n-1} \Bigg\{ \Bigg[ \frac{(z\zeta\omega^{2k} + \overline{z}\overline{\zeta}\omega^{-2k})^3}{36|\zeta|^6} + \frac{(z\zeta\omega^{2k} + \overline{z}\overline{\zeta}\omega^{-2k})}{12|\zeta|^2}(|\zeta|^2 + |z|^2)^2 \\ &\quad - \frac{(z^2\zeta^2\omega^{4k} + \overline{z}^2\overline{\zeta}^2\omega^{-4k})}{12|\zeta|^4}(|\zeta|^2 + |z|^2) \Bigg] \log|\omega^{-2k} - z\zeta|^2 \\ &\quad - \Bigg[ \frac{(z\overline{\zeta}\omega^{2k} + \overline{z}\zeta\omega^{-2k})^3}{36|\zeta|^6} + \frac{(z\overline{\zeta}\omega^{2k} + \overline{z}\zeta\omega^{-2k})}{12|\zeta|^2}(|\zeta|^2 + |z|^2)^2 \\ &\quad - \frac{(\overline{z}^2\zeta^2\omega^{-4k} + z^2\overline{\zeta}^2\omega^{4k})}{12|\zeta|^4}(|\zeta|^2 + |z|^2) \Bigg] \log|\omega^{2k} - \overline{z}\zeta|^2 \\ &\quad + \Bigg[ \frac{1}{36}(z\overline{\zeta}\omega^{2k} + \overline{z}\zeta\omega^{-2k})^3 + \frac{1}{12}(z\overline{\zeta}\omega^{2k} + \overline{z}\zeta\omega^{-2k})(|\zeta|^2 + |z|^2)^2 \\ &\quad - \frac{1}{12}(\overline{z}^2\zeta^2\omega^{-4k} + z^2\overline{\zeta}^2\omega^{4k})(|\zeta|^2 + |z|^2) \Bigg] \log|\zeta - z\omega^{2k}|^2 \end{aligned}$$

$$
-\left[\frac{1}{36}(z\zeta\omega^{2k}+\overline{z}\overline{\zeta}\omega^{-2k})^3+\frac{1}{12}(z\zeta\omega^{2k}+\overline{z}\overline{\zeta}\omega^{-2k})(|\zeta|^2+|z|^2)^2 \right.
$$

$$
\left. -\frac{1}{12}(\overline{z}^2\overline{\zeta}^2\omega^{-4k}+z^2\zeta^2\omega^{4k})(|\zeta|^2+|z|^2)\right]\log|\overline{\zeta}-z\omega^{2k}|^2\Bigg\}. \tag{6.7}
$$

Through computation, we obtain

$$
(\partial_z\partial_{\overline{z}})^3[h_4(z,\zeta)-F_4(z,\zeta)]=0, \quad z,\ \zeta\in\Omega, \tag{6.8}
$$

and

$$
\begin{cases}
h_4(z,\zeta)-F_4(z,\zeta)=0, & z\in[0,1]\cup[\omega,0],\\
\partial_z\partial_{\overline{z}}[h_4(z,\zeta)-F_4(z,\zeta)]=0, & z\in[0,1]\cup[\omega,0],\\
(\partial_z\partial_{\overline{z}})^2[h_4(z,\zeta)-F_4(z,\zeta)]=0, & z\in[0,1]\cup[\omega,0].
\end{cases} \tag{6.9}
$$

Also, we have

$$
h_4(z,\zeta)-F_4(z,\zeta)=2\sum_{k=0}^{n-1}\mathrm{Re}\left\{\Xi_3(z\omega^{2k},\zeta)-\Xi_3(\overline{z}\omega^{-2k},\zeta)\right\}, \quad z\in L, \tag{6.10}
$$

$$
\partial_z\partial_{\overline{z}}\left[h_4(z,\zeta)-F_4(z,\zeta)\right]=2\sum_{k=0}^{n-1}\mathrm{Re}\left\{\Xi_4(z\omega^{2k},\zeta)-\Xi_4(\overline{z}\omega^{-2k},\zeta)\right\},\ z\in L, \tag{6.11}
$$

$$
(\partial_z\partial_{\overline{z}})^2\left[h_4(z,\zeta)-F_4(z,\zeta)\right]=2\sum_{k=0}^{n-1}\mathrm{Re}\left\{\Xi_5(z\omega^{2k},\zeta)-\Xi_5(\overline{z}\omega^{-2k},\zeta)\right\},\ z\in L, \tag{6.12}
$$

with

$$
\Xi_3(z,\zeta)=\left\{\frac{|\zeta|^6-1}{36|\zeta|^6}\quad(z^3\zeta^3+3z\zeta|\zeta|^2)+\frac{(|\zeta|^4-1)(1+|\zeta|^2)}{12|\zeta|^2}z\zeta \right.
$$

$$
\left. +\frac{z^2(1-|\zeta|^4)(1+|\zeta|^2)}{12\overline{\zeta}^2}\right\}\log|1-z\zeta|^2, \tag{6.13}
$$

$$
\Xi_4(z,\zeta)=\frac{|\zeta|^4-1}{12|\zeta|^4}\left[(6|\zeta|^2+2)z\zeta-3z^2\zeta^2\right]\log|1-z\zeta|^2+\frac{|\zeta|^4-1}{6|\zeta|^2}\left[z\zeta-\frac{z^2\zeta^2}{|\zeta|^2}\right]
$$

$$
+\frac{|\zeta|^4-1}{12|\zeta|^2}\frac{1}{1-z\zeta}\left[-4(1+|\zeta|^2)+2\overline{z}\overline{\zeta}\right], \tag{6.14}
$$

$$\Xi_5(z,\zeta) = \frac{z(|\zeta|^2-1)}{\overline{\zeta}} \log|1-z\zeta|^2 + \frac{1}{6(1-z\zeta)^2}\left[\frac{\overline{z}(1-|\zeta|^2)}{\zeta} + \frac{2}{|\zeta|^2} - |\zeta|^2 - 1\right]$$
$$+\frac{1}{6(1-z\zeta)}\left[\frac{\overline{z}(5|\zeta|^2-6)}{\zeta} + \frac{4}{|\zeta|^2} - 8|\zeta|^2 + 5\right] + \frac{z(6|\zeta|^2-5)}{6\overline{\zeta}}. \tag{6.15}$$

Then, the next theorem holds.

**Theorem 6.1.1.** *For* $z,\ \zeta \in \Omega$,

$$h_4(z,\zeta) = F_4(z,\zeta) - \text{Re}\ \left\{\frac{1}{2\pi i}\int\limits_{|\widetilde{\zeta}|=1}\left[\partial_{\nu_{\widetilde{\zeta}}} G_1(z,\widetilde{\zeta})\Xi_3(\widetilde{\zeta},\zeta)\right.\right.$$
$$\left.\left.+\partial_{\nu_{\widetilde{\zeta}}}\widehat{G}_2(z,\widetilde{\zeta})\Xi_4(\widetilde{\zeta},\zeta) + \partial_{\nu_{\widetilde{\zeta}}}\widehat{G}_3(z,\widetilde{\zeta})\Xi_5(\widetilde{\zeta},\zeta)\right]\frac{d\widetilde{\zeta}}{\widetilde{\zeta}}\right\},$$

*where* $F_4$, $\partial_{\nu_{\widetilde{\zeta}}} G_1$, $\partial_{\nu_{\widetilde{\zeta}}}\widehat{G}_2$, $\Xi_i$ $(i = 3,4,5)$ *are given in* (6.7), (5.8), (5.9) *and* (6.13)-(6.15) *respectively. Moreover,* $B_0$ *is defined by* (4.33) *and for* $z \in L$,

$$\partial_{\nu_z}\widehat{G}_3(z,\zeta) = \frac{|\zeta|^4-1}{|\zeta|^2}(\overline{\zeta}-\zeta)(\overline{z}-z)B_0 + 2(1-|\zeta|^2)$$
$$\times\sum_{k=0}^{n-1}\sum_{l=0}^{\infty}\frac{1}{(l+1)^2}\left[(\overline{z}\overline{\zeta}\omega^{-2k})^l + (z\zeta\omega^{2k})^l - (z\overline{\zeta}\omega^{2k})^l - (\overline{z}\zeta\omega^{-2k})^l\right]$$
$$+(|\zeta|^4-1)\sum_{k=0}^{n-1}\left\{\frac{\log(1-z\zeta\omega^{2k})}{(z\zeta\omega^{2k})^2} + \frac{\log(1-\overline{z}\overline{\zeta}\omega^{-2k})}{(\overline{z}\overline{\zeta}\omega^{-2k})^2} - \frac{\log(1-\overline{z}\zeta\omega^{-2k})}{(\overline{z}\zeta\omega^{-2k})^2}\right.$$
$$-\frac{\log(1-z\overline{\zeta}\omega^{2k})}{(z\overline{\zeta}\omega^{2k})^2} - \frac{\log(1-z\zeta\omega^{2k})}{z\zeta\omega^{2k}} - \frac{\log(1-\overline{z}\overline{\zeta}\omega^{-2k})}{\overline{z}\overline{\zeta}\omega^{-2k}}$$
$$\left.+\frac{\log(1-\overline{z}\zeta\omega^{-2k})}{\overline{z}\zeta\omega^{-2k}} + \frac{\log(1-z\overline{\zeta}\omega^{2k})}{z\overline{\zeta}\omega^{2k}}\right\}.$$

**Proof:** Let $\widetilde{F_4}(z,\zeta) = h_4(z,\zeta) - F_4(z,\zeta)$, thus from (6.8), (6.9) and Theorem 5.3.2, we obtain

$$h_4(z,\zeta) = F_4(z,\zeta) - \frac{1}{4\pi i}\int\limits_L\left[\partial_{\nu_{\widetilde{\zeta}}} G_1(z,\widetilde{\zeta})\widetilde{F_4}(\widetilde{\zeta},\zeta)\right.$$
$$\left.+\partial_{\nu_{\widetilde{\zeta}}}\widehat{G}_2(z,\widetilde{\zeta})\partial_{\widetilde{\zeta}}\partial_{\overline{\widetilde{\zeta}}}\widetilde{F_4}(\widetilde{\zeta},\zeta) + \partial_{\nu_{\widetilde{\zeta}}}\widehat{G}_3(z,\widetilde{\zeta})(\partial_{\widetilde{\zeta}}\partial_{\overline{\widetilde{\zeta}}})^2\widetilde{F_4}(\widetilde{\zeta},\zeta)\right]\frac{d\widetilde{\zeta}}{\widetilde{\zeta}}. \tag{6.16}$$

Then from (6.10)-(6.12), the boundary integral in (6.16) can be represented as

$$\begin{aligned}
&\mathrm{Re}\Bigg\{\frac{1}{2\pi i}\int\limits_{L}\Big\{\partial_{\nu_{\widetilde{\zeta}}}G_1(z,\widetilde{\zeta})\left[\Xi_3(\widetilde{\zeta}\omega^{2k},\zeta)-\Xi_3(\overline{\widetilde{\zeta}}\omega^{-2k},\zeta)\right] \\
&\qquad+\partial_{\nu_{\widetilde{\zeta}}}\widehat{G}_2(z,\widetilde{\zeta})\left[\Xi_4(\widetilde{\zeta}\omega^{2k},\zeta)-\Xi_4(\overline{\widetilde{\zeta}}\omega^{-2k},\zeta)\right] \\
&\qquad+\partial_{\nu_{\widetilde{\zeta}}}\widehat{G}_3(z,\widetilde{\zeta})\left[\Xi_5(\widetilde{\zeta}\omega^{2k},\zeta)-\Xi_5(\overline{\widetilde{\zeta}}\omega^{-2k},\zeta)\right]\Big\}\frac{\mathrm{d}\widetilde{\zeta}}{\widetilde{\zeta}}\Bigg\} \\
&=\mathrm{Re}\Bigg\{\frac{1}{2\pi i}\int\limits_{|\widetilde{\zeta}|=1}\Big[\partial_{\nu_{\widetilde{\zeta}}}G_1(z,\widetilde{\zeta})\Xi_3(\widetilde{\zeta},\zeta)+\partial_{\nu_{\widetilde{\zeta}}}\widehat{G}_2(z,\widetilde{\zeta})\Xi_4(\widetilde{\zeta},\zeta) \\
&\qquad\qquad+\partial_{\nu_{\widetilde{\zeta}}}\widehat{G}_3(z,\widetilde{\zeta})\Xi_5(\widetilde{\zeta},\zeta)\Big]\frac{\mathrm{d}\widetilde{\zeta}}{\widetilde{\zeta}}\Bigg\}.
\end{aligned}$$

Thus, the proof is completed.

Suppose $C_0=\sum\limits_{k=0}^{n-1}\omega^{4k}$, then we have

$$C_0=\begin{cases}1, & n=1,\\ 2, & n=2,\\ 0, & n>2.\end{cases}\tag{6.17}$$

Next, the following three lemmas are needed.

**Lemma 6.1.1.** *For* $z,\ \zeta\in\Omega$,

$$\begin{aligned}
&H_1(z,\zeta)=-\mathrm{Re}\Bigg\{\frac{1}{2\pi i}\int\limits_{|\widetilde{\zeta}|=1}\partial_{\nu_{\widetilde{\zeta}}}G_1(z,\widetilde{\zeta})\Xi_3(\widetilde{\zeta},\zeta)\frac{\mathrm{d}\widetilde{\zeta}}{\widetilde{\zeta}}\Bigg\} \\
&=\frac{(1-|z|^4)(1-|\zeta|^6)}{36|z|^4|\zeta|^4}(\overline{\zeta}^2-\zeta^2)(\overline{z}^2-z^2)C_0 \\
&\quad+\frac{(1-|z|^2)(1-|\zeta|^2)}{72|\zeta|^2|z|^2}(5|\zeta|^4+11|\zeta|^2+5)(\overline{\zeta}-\zeta)(\overline{z}-z)B_0 \\
&+\frac{1-|\zeta|^6}{36|\zeta|^6}\sum_{k=0}^{n-1}\Bigg\{\left[(\overline{z}\zeta\omega^{-2k})^3+\frac{\overline{\zeta}^3}{(\overline{z}\omega^{-2k})^3}\right]\log(1-\overline{z}\zeta\omega^{-2k}) \\
&+\left[(z\overline{\zeta}\omega^{2k})^3+\frac{\zeta^3}{(z\omega^{2k})^3}\right]\log(1-z\overline{\zeta}\omega^{2k})-\left[(z\zeta\omega^{2k})^3+\frac{\overline{\zeta}^3}{(z\omega^{2k})^3}\right]\log(1-z\zeta\omega^{2k})
\end{aligned}$$

$$
- \left[ (\overline{z}\overline{\zeta}\omega^{-2k})^3 + \frac{\zeta^3}{(\overline{z}\omega^{-2k})^3} \right] \log(1 - \overline{z}\overline{\zeta}\omega^{-2k}) \Bigg\}
$$

$$
+\frac{1}{12}\left[(|\zeta|^2+1)^2(1-|\zeta|^2)+\frac{1-|\zeta|^6}{|\zeta|^2}\right]\sum_{k=0}^{n-1}\Bigg\{\left[\frac{1}{\overline{z}\zeta\omega^{-2k}}+\frac{\overline{z}\zeta\omega^{-2k}}{|\zeta|^2}\right]\log(1-\overline{z}\zeta\omega^{-2k})
$$

$$
+\left[\frac{1}{z\overline{\zeta}\omega^{2k}}+\frac{z\overline{\zeta}\omega^{2k}}{|\zeta|^2}\right]\log(1-z\overline{\zeta}\omega^{2k})-\left[\frac{1}{\overline{z}\overline{\zeta}\omega^{-2k}}+\frac{\overline{z}\overline{\zeta}\omega^{-2k}}{|\zeta|^2}\right]\log(1-\overline{z}\overline{\zeta}\omega^{-2k})
$$

$$
-\left[\frac{1}{z\zeta\omega^{2k}}+\frac{z\zeta\omega^{2k}}{|\zeta|^2}\right]\log(1-z\zeta\omega^{2k})\Bigg\}
$$

$$
-\frac{1}{12}(1-|\zeta|^4)(1+|\zeta|^2)\sum_{k=0}^{n-1}\Bigg\{\left[\frac{\omega^{4k}}{(\overline{z}\zeta)^2}+\frac{(\overline{z}\zeta)^2}{|\zeta|^4\omega^{4k}}\right]\log(1-\overline{z}\zeta\omega^{-2k})
$$

$$
+\left[\frac{\omega^{-4k}}{(z\overline{\zeta})^2}+\frac{(z\overline{\zeta})^2}{|\zeta|^4\omega^{-4k}}\right]\log(1-z\overline{\zeta}\omega^{2k})-\left[\frac{\omega^{4k}}{(\overline{z}\overline{\zeta})^2}+\frac{(\overline{z}\overline{\zeta})^2}{|\zeta|^4\omega^{4k}}\right]\log(1-\overline{z}\overline{\zeta}\omega^{-2k})
$$

$$
-\left[\frac{\omega^{-4k}}{(z\zeta)^2}+\frac{(z\zeta)^2}{|\zeta|^4\omega^{-4k}}\right]\log(1-z\zeta\omega^{2k})\Bigg\},
$$

*with $B_0$, $C_0$ given by* (4.33) *and* (6.17) *respectively.*

**Lemma 6.1.2.** *For $z,\ \zeta \in \Omega$,*

$$
H_2(z,\zeta) = -\mathrm{Re}\left\{\frac{1}{2\pi i}\int_{|\widetilde{\zeta}|=1}\partial_{\nu_{\widetilde{\zeta}}}\widehat{G}_2(z,\widetilde{\zeta})\Xi_4(\widetilde{\zeta},\zeta)\frac{\mathrm{d}\widetilde{\zeta}}{\widetilde{\zeta}}\right\}
$$

$$
= \frac{(1-|z|^2)(1-|\zeta|^4)(2|z|^2+3)}{12|z|^2|\zeta|^2}(\overline{\zeta}-\zeta)(\overline{z}-z)B_0
$$

$$
-\frac{(1-|z|^2)(1-|\zeta|^4)}{36|\zeta|^4}(\overline{\zeta}^2-\zeta^2)(\overline{z}^2-z^2)C_0-\frac{(|\zeta|^4-1)(|z|^2-1)(6|\zeta|^2+2)}{12|\zeta|^2}
$$

$$
\times\sum_{k=0}^{n-1}\sum_{l=0}^{\infty}\frac{1}{(l+1)^2}\left[(\overline{z}\overline{\zeta}\omega^{-2k})^l+(z\zeta\omega^{2k})^l-(z\overline{\zeta}\omega^{2k})^l-(\overline{z}\zeta\omega^{-2k})^l\right]
$$

$$
+\frac{(|\zeta|^4-1)(|z|^2-1)}{12|\zeta|^2}\sum_{k=0}^{n-1}\Bigg\{\Bigg[3z\zeta\omega^{2k}+\frac{\omega^{-2k}}{z\zeta}+\frac{z\zeta}{|\zeta|^2\omega^{-2k}}
$$

$$
-\frac{z^2\zeta^2\omega^{4k}}{|\zeta|^2}-\frac{|\zeta|^2}{z\zeta\omega^{2k}}+\frac{3|\zeta|^2}{(z\zeta\omega^{2k})^2}\Bigg]\log(1-z\zeta\omega^{2k})
$$

$$+\left[3\overline{z}\overline{\zeta}\omega^{-2k}+\frac{\omega^{2k}}{\overline{z}\overline{\zeta}}+\frac{\overline{z}\overline{\zeta}}{|\zeta|^2\omega^{2k}}\right.$$

$$\left.-\frac{\overline{z}^2\overline{\zeta}^2\omega^{-4k}}{|\zeta|^2}-\frac{|\zeta|^2}{\overline{z}\overline{\zeta}\omega^{-2k}}+\frac{3|\zeta|^2}{(\overline{z}\overline{\zeta}\omega^{-2k})^2}\right]\log(1-\overline{z}\overline{\zeta}\omega^{-2k})$$

$$-\left[3z\overline{\zeta}\omega^{2k}+\frac{\omega^{-2k}}{z\overline{\zeta}}+\frac{z\overline{\zeta}}{|\zeta|^2\omega^{-2k}}\right.$$

$$\left.\left.-\frac{z^2\overline{\zeta}^2\omega^{4k}}{|\zeta|^2}-\frac{|\zeta|^2}{z\overline{\zeta}\omega^{2k}}+\frac{3|\zeta|^2}{(z\overline{\zeta}\omega^{2k})^2}\right]\right\}\log(1-z\overline{\zeta}\omega^{2k})$$

$$-\left[3\overline{z}\zeta\omega^{-2k}+\frac{\omega^{2k}}{\overline{z}\zeta}+\frac{\overline{z}\zeta}{|\zeta|^2\omega^{2k}}\right.$$

$$\left.\left.-\frac{\overline{z}^2\zeta^2\omega^{-4k}}{|\zeta|^2}-\frac{|\zeta|^2}{\overline{z}\zeta\omega^{-2k}}+\frac{3|\zeta|^2}{(\overline{z}\zeta\omega^{-2k})^2}\right]\log(1-\overline{z}\zeta\omega^{-2k})\right\},$$

*where* $B_0$, $C_0$ *are given by* (4.33) *and* (6.17) *respectively.*

**Lemma 6.1.3.** *For* $z,\ \zeta\in\Omega$,

$$H_3(z,\zeta)=-\mathrm{Re}\left\{\frac{1}{2\pi i}\int\limits_{|\widetilde{\zeta}|=1}\partial_{\nu_{\widetilde{\zeta}}}\widehat{G}_3(z,\widetilde{\zeta})\Xi_5(\widetilde{\zeta},\zeta)\frac{d\widetilde{\zeta}}{\widetilde{\zeta}}\right\}$$

$$=\left[\frac{(1-|z|^2)(1-|\zeta|^2)(|z|^2+2)}{6|z|^2|\zeta|^2}+\frac{5(1-|z|^4)(1-|\zeta|^2)}{72|\zeta|^2}\right](\overline{\zeta}-\zeta)(\overline{z}-z)B_0$$

$$+(|\zeta|^2-1)(|z|^2-1)\sum_{k=0}^{n-1}\sum_{l=0}^{\infty}\frac{1}{(l+1)^3}\left[(\overline{z}\overline{\zeta}\omega^{-2k})^l+(z\zeta\omega^{2k})^l\right.$$

$$\left.-(z\overline{\zeta}\omega^{2k})^l-(\overline{z}\zeta\omega^{-2k})^l\right]$$

$$-\frac{1}{6}\left[\frac{|z|^2-1}{|\zeta|^2}(1+3|\zeta|^2-4|\zeta|^4)+3(|\zeta|^2-1)(|z|^4-1)\right]$$

$$\times\sum_{k=0}^{n-1}\sum_{l=0}^{\infty}\frac{1}{(l+1)^2}\left[(\overline{z}\overline{\zeta}\omega^{-2k})^l+(z\zeta\omega^{2k})^l-(z\overline{\zeta}\omega^{2k})^l-(\overline{z}\zeta\omega^{-2k})^l\right]$$

$$+\sum_{k=0}^{n-1}\left\{\frac{1}{3}(|\zeta|^2-1)(|z|^4-1)\left[\frac{\log(1-z\zeta\omega^{2k})}{(z\zeta\omega^{2k})^2}+\frac{\log(1-\overline{z}\overline{\zeta}\omega^{-2k})}{(\overline{z}\overline{\zeta}\omega^{-2k})^2}\right.\right.$$

$$\left.-\frac{\log(1-\overline{z}\zeta\omega^{-2k})}{(\overline{z}\zeta\omega^{-2k})^2}-\frac{\log(1-z\overline{\zeta}\omega^{2k})}{(z\overline{\zeta}\omega^{2k})^2}\right]-\frac{(|\zeta|^2-1)(|z|^2-1)}{12|\zeta|^2}(2-|z|^2)$$

$$\times\Bigg[z\zeta\omega^{2k}\log(1-z\zeta\omega^{2k})+\overline{z}\overline{\zeta}\omega^{-2k}\log(1-\overline{z}\overline{\zeta}\omega^{-2k})$$

$$-\overline{z}\zeta\omega^{-2k}\log(1-\overline{z}\zeta\omega^{-2k})-z\overline{\zeta}\omega^{2k}\log(1-z\overline{\zeta}\omega^{2k})\Bigg]$$

$$-\left[\frac{|z|^2(|\zeta|^2-1)(|z|^2-1)}{12|\zeta|^2}(-4|\zeta|^2-1)+\frac{1}{2}(|\zeta|^2-1)(|z|^4-1)\right]$$

$$\times\Bigg[\frac{\log(1-z\zeta\omega^{2k})}{z\zeta\omega^{2k}}+\frac{\log(1-\overline{z}\overline{\zeta}\omega^{-2k})}{\overline{z}\overline{\zeta}\omega^{-2k}}$$

$$-\frac{\log(1-\overline{z}\zeta\omega^{-2k})}{\overline{z}\zeta\omega^{-2k}}-\frac{\log(1-z\overline{\zeta}\omega^{2k})}{z\overline{\zeta}\omega^{2k}}\Bigg]\Bigg\},$$

*with* $B_0$ *given by* (4.33).

Then from Theorem 6.1.1,

$$h_4(z,\zeta)=F_4(z,\zeta)+H_1(z,\zeta)+H_2(z,\zeta)+H_3(z,\zeta), \tag{6.18}$$

where $F_4$, $H_i$ $(i=1,2,3)$ are defined by (6.7) and Lemmas 6.1.1-6.1.3 respectively.

Therefore, by (6.2) and (6.18), $\widehat{G}_4$ is represented as

$$\widehat{G}_4(z,\zeta)=\frac{1}{36}(|z|^2-1)(|\zeta|^2-1)(\overline{z}^2-z^2)(\overline{\zeta}^2-\zeta^2)C_0$$

$$+\frac{1}{72}(|z|^4-1)(|\zeta|^4-1)(\overline{z}-z)(\overline{\zeta}-\zeta)B_0$$

$$-\frac{1}{12}(|z|^2-1)(|\zeta|^2-1)(|\zeta|^2+|z|^2)(\overline{z}-z)(\overline{\zeta}-\zeta)B_0$$

$$+(|z|^2-1)(|\zeta|^2-1)\sum_{k=0}^{n-1}\sum_{l=1}^{\infty}\frac{1}{(l+1)^3}\Bigg[(z\zeta\omega^{2k})^l+(\overline{z}\overline{\zeta}\omega^{-2k})^l$$

$$-(z\overline{\zeta}\omega^{2k})^l-(\overline{z}\zeta\omega^{-2k})^l\Bigg]$$

$$-\frac{1}{2}(|z|^2-1)(|\zeta|^2-1)(|z|^2+|\zeta|^2+1)\sum_{k=0}^{n-1}\sum_{l=1}^{\infty}\frac{1}{(l+1)^2}\Bigg[(z\zeta\omega^{2k})^l+(\overline{z}\overline{\zeta}\omega^{-2k})^l$$

$$-(z\overline{\zeta}\omega^{2k})^l-(\overline{z}\zeta\omega^{-2k})^l\Bigg]$$

$$
\begin{aligned}
&+\sum_{k=0}^{n-1}\left\{\frac{1}{36}\left|\zeta-z\omega^{2k}\right|^6\log\left|\frac{1-z\overline{\zeta}\omega^{2k}}{\zeta-z\omega^{2k}}\right|^2+\frac{1}{36}\left|\zeta-\overline{z}\omega^{-2k}\right|^6\log\left|\frac{\zeta-\overline{z}\omega^{-2k}}{1-z\zeta\omega^{2k}}\right|^2\right.\\
&\quad-\frac{1}{36}(|z|^6-1)(|\zeta|^6-1)\left\{\frac{1}{2!}\left(\frac{1}{z\zeta\omega^{2k}}+\frac{1}{\overline{z}\overline{\zeta}\omega^{-2k}}-\frac{1}{\overline{z}\zeta\omega^{-2k}}-\frac{1}{z\overline{\zeta}\omega^{2k}}\right)\right.\\
&\quad\quad+\frac{1}{(z\zeta\omega^{2k})^2}+\frac{1}{(\overline{z}\overline{\zeta}\omega^{-2k})^2}-\frac{1}{(\overline{z}\zeta\omega^{-2k})^2}-\frac{1}{(z\overline{\zeta}\omega^{2k})^2}+\frac{\log(1-z\zeta\omega^{2k})}{(z\zeta\omega^{2k})^3}\\
&\quad\quad\quad\left.+\frac{\log(1-\overline{z}\overline{\zeta}\omega^{-2k})}{(\overline{z}\overline{\zeta}\omega^{-2k})^3}-\frac{\log(1-\overline{z}\zeta\omega^{-2k})}{(\overline{z}\zeta\omega^{-2k})^3}-\frac{\log(1-z\overline{\zeta}\omega^{2k})}{(z\overline{\zeta}\omega^{2k})^3}\right\}\\
&+\frac{1}{12}\Big[(|z|^6-1)(|\zeta|^4-1)+(|z|^4-1)(|\zeta|^6-1)\\
&\qquad\qquad+3(|z|^2-1)(|\zeta|^4-1)+3(|z|^4-1)(|\zeta|^2-1)\Big]\\
&\times\left\{\frac{\log(1-z\zeta\omega^{2k})}{(z\zeta\omega^{2k})^2}+\frac{\log(1-\overline{z}\overline{\zeta}\omega^{-2k})}{(\overline{z}\overline{\zeta}\omega^{-2k})^2}-\frac{\log(1-\overline{z}\zeta\omega^{-2k})}{(\overline{z}\zeta\omega^{-2k})^2}-\frac{\log(1-z\overline{\zeta}\omega^{2k})}{(z\overline{\zeta}\omega^{2k})^2}\right.\\
&\quad\left.+\frac{1}{z\zeta\omega^{2k}}+\frac{1}{\overline{z}\overline{\zeta}\omega^{-2k}}-\frac{1}{\overline{z}\zeta\omega^{-2k}}-\frac{1}{z\overline{\zeta}\omega^{2k}}\right\}\\
&-\frac{1}{12}\Big[(|z|^6-1)(|\zeta|^2-1)+(|z|^2-1)(|\zeta|^6-1)\\
&\qquad\qquad+3(|z|^4-1)(|\zeta|^4-1)+3(|z|^2-1)(|\zeta|^2-1)\Big]\\
&\quad\times\left[\frac{\log(1-z\zeta\omega^{2k})}{z\zeta\omega^{2k}}+\frac{\log(1-\overline{z}\overline{\zeta}\omega^{-2k})}{\overline{z}\overline{\zeta}\omega^{-2k}}\right.\\
&\qquad\qquad\left.\left.-\frac{\log(1-\overline{z}\zeta\omega^{-2k})}{\overline{z}\zeta\omega^{-2k}}-\frac{\log(1-z\overline{\zeta}\omega^{2k})}{z\overline{\zeta}\omega^{2k}}\right]\right\},
\end{aligned}
\tag{6.19}
$$

where $B_0,\ C_0$ are defined by (4.33) and (6.17) respectively. We also can verify that the expression (6.19) satisfies the properties

$$(\partial_z\partial_{\overline{z}})^3\widehat{G}_4(z,\zeta)=G_1(z,\zeta),\qquad z,\ \zeta\in\Omega,$$

$$(\partial_z\partial_{\overline{z}})^{j-1}\widehat{G}_4(z,\zeta)=0,\qquad z\in\partial\Omega,\ j=1,2,3,$$

which implies that expression (6.19) is just the desired tetra-harmonic Green function for the domain $\Omega$.

Furthermore, the following representation formula and the statement about the Dirichlet problem hold.

**Theorem 6.1.2.** *Any* $w \in C^8(\Omega;\mathbb{C}) \cap C^7(\overline{\Omega};\mathbb{C})$ *can be represented as*

$$w(z) = -\sum_{\mu=1}^{4} \frac{1}{4\pi} \int_{\partial\Omega} \partial_{\nu_\zeta} \widehat{G}_\mu(z,\zeta)(\partial_\zeta \partial_{\overline{\zeta}})^{\mu-1} w(\zeta) \mathrm{d}s_\zeta - \frac{1}{\pi} \int_{\Omega} \widehat{G}_4(z,\zeta)(\partial_\zeta \partial_{\overline{\zeta}})^4 w(\zeta) \mathrm{d}\xi \mathrm{d}\eta,$$

*where* $\widehat{G}_1(z,\zeta) = G_1(z,\zeta)$ *and* $\widehat{G}_\mu(z,\zeta)$ $(\mu = 2,3,4)$ *are the harmonic Green, the biharmonic Green, the triharmonic Green and the tetra-harmonic Green functions for* $\Omega$ *respectively.*

**Theorem 6.1.3.** *The tetra-harmonic Dirichlet problem*

$$\begin{cases} (\partial_z \partial_{\overline{z}})^4 w(z) = f \quad in \quad \Omega, \\ w = \gamma_0, \quad \partial_z \partial_{\overline{z}} w = \gamma_1, \quad (\partial_z \partial_{\overline{z}})^2 w = \gamma_2, \quad (\partial_z \partial_{\overline{z}})^3 w = \gamma_3 \quad on \quad \partial\Omega, \end{cases}$$

*with* $f \in L_p(\Omega;\mathbb{C}),\ p > 2,\ \gamma_0,\ \gamma_1,\ \gamma_2,\ \gamma_3 \in C(\partial\Omega;\mathbb{C})$ *is uniquely solvable by*

$$\begin{aligned} w(z) = -\frac{1}{4\pi} \int_{\partial\Omega} \Big[ & \partial_{\nu_\zeta} G_1(z,\zeta)\gamma_0(\zeta) + \partial_{\nu_\zeta} \widehat{G}_2(z,\zeta)\gamma_1(\zeta) + \partial_{\nu_\zeta} \widehat{G}_3(z,\zeta)\gamma_2(\zeta) \\ & + \partial_{\nu_\zeta} \widehat{G}_4(z,\zeta)\gamma_3(\zeta) \Big] \mathrm{d}s_\zeta - \frac{1}{\pi} \int_{\Omega} \widehat{G}_4(z,\zeta) f(\zeta) \mathrm{d}\xi \mathrm{d}\eta, \quad z \in \Omega, \end{aligned}$$

*where* $G_1(z,\zeta),\ \widehat{G}_2(z,\zeta),\ \widehat{G}_3(z,\zeta),\ \widehat{G}_4(z,\zeta)$ *are given in* (4.1), (4.16), (5.21) *and* (6.19) *respectively.*

## 6.2 Tetra-harmonic Neumann Function

In the following, we want to find a tetra-harmonic Neumann function $N_4(z,\zeta)$ which satisfies,

1. $N_4(z,\zeta)$ is tetra-harmonic in $\Omega \backslash \{\zeta\}$ with respect to $z$,
2. $N_4(z,\zeta) + \frac{1}{36}|\zeta - z|^6 \log|\zeta - z|^2$ is tetra-harmonic in $\Omega$,
3. $\partial_z \partial_{\overline{z}} N_4(z,\zeta) = N_3(z,\zeta),\ z,\zeta \in \Omega$,

$$4.\ \partial_{\nu_z} N_4(z,\zeta) = \begin{cases} \dfrac{4n}{\pi}\displaystyle\int_\Omega N_3(\zeta,\widetilde{\zeta})\mathrm{d}\widetilde{\xi}\mathrm{d}\widetilde{\eta} \\ \quad = 4n\left[\dfrac{1}{6}(|\zeta|^2-1)^2 - \dfrac{1}{3}(|\zeta|^2-1) - \dfrac{1}{36}(|\zeta|^2-1)^3\right], \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad z \in L\backslash\{1,\omega\}, \\ 0, \qquad\qquad\qquad\qquad\qquad z\in(0,1)\cup(\omega,0), \end{cases}$$

$$5. \int_L N_4(z,\zeta)\frac{\mathrm{d}z}{z} = 0 \text{ for } \zeta\in\Omega,$$

where $N_3(z,\zeta)$ is given in (5.44), and the boundary behavior 4 above follows from applying Theorem 5.3.4 to the function $\frac{1}{36}(|z|^2-1)^3$.

We also have

$$N_4(z,\zeta) = -\frac{1}{\pi}\int_\Omega N_1(z,\widetilde{\zeta})N_3(\widetilde{\zeta},\zeta)\mathrm{d}\widetilde{\xi}\mathrm{d}\widetilde{\eta}, \quad z,\ \zeta\in\Omega,$$

where $N_1,\ N_3$ are given by (4.17) and (5.44) respectively. Then we firstly transform $N_4(z,\zeta)$ into

$$N_4(z,\zeta) = \frac{1}{36}|\zeta - z|^6 N_1(z,\zeta) + \widehat{h}_4(z,\zeta), \quad z,\zeta\in\Omega, \tag{6.20}$$

where $\widehat{h}_4(z,\zeta)$ is a tetra-harmonic function in $\Omega$. Then we have

$$(\partial_z\partial_{\overline{z}})^3\widehat{h}_4(z,\zeta) = f_4(z,\zeta), \quad z,\ \zeta\in\Omega, \tag{6.21}$$

$$\begin{aligned}\partial_z\partial_{\overline{z}}\widehat{h}_4(z,\zeta) = N_3(z,\zeta) - \frac{1}{4}|\zeta-z|^4 N_1 + \frac{1}{12}|\zeta-z|^4\Big[(\overline{\zeta}-\overline{z})\partial_{\overline{z}}N_1 \\ +(\zeta-z)\partial_z N_1\Big], \quad z,\ \zeta\in\Omega,\end{aligned} \tag{6.22}$$

$$\begin{aligned}(\partial_z\partial_{\overline{z}})^2\widehat{h}_4(z,\zeta) &= N_2(z,\zeta) - |\zeta-z|^2 N_1 \\ &+\frac{1}{3}|\zeta-z|^2\mathrm{Re}\Big\{6(\zeta-z)\partial_z N_1 - (\zeta-z)^2\partial_z^2 N_1\Big\},\ z,\ \zeta\in\Omega,\end{aligned} \tag{6.23}$$

where

$$
\begin{aligned}
f_4(z,\zeta) &= 2\mathrm{Re}\Big\{3(\zeta-z)\partial_z N_1(z,\zeta) - \frac{3}{2}(\zeta-z)^2\partial_z^2 N_1(z,\zeta) \\
&\qquad + \frac{1}{6}(\zeta-z)^3\partial_z^3 N_1(z,\zeta)\Big\} \\
&= \frac{44}{3}n + 2\sum_{k=0}^{n-1}\mathrm{Re}\Big\{\frac{|\zeta|^2-\omega^{-2k}}{\omega^{-2k}-z\overline{\zeta}} + \frac{\zeta\omega^{2k}-\overline{\zeta}}{\overline{\zeta}-z\omega^{2k}} + \frac{\zeta^2-\omega^{-2k}}{\omega^{-2k}-z\zeta} + \frac{\zeta\omega^{2k}-\zeta}{\zeta-z\omega^{2k}} \\
&+\frac{1}{3}\left[\frac{(|\zeta|^2-\omega^{-2k})^3}{(\omega^{-2k}-z\overline{\zeta})^3} + \frac{(\zeta\omega^{2k}-\overline{\zeta})^3}{(\overline{\zeta}-z\omega^{2k})^3} + \frac{(\zeta^2-\omega^{-2k})^3}{(\omega^{-2k}-z\zeta)^3} + \frac{(\zeta\omega^{2k}-\zeta)^3}{(\zeta-z\omega^{2k})^3}\right] \\
&-\frac{1}{2}\left[\frac{(|\zeta|^2-\omega^{-2k})^2}{(\omega^{-2k}-z\overline{\zeta})^2} + \frac{(\zeta\omega^{2k}-\overline{\zeta})^2}{(\overline{\zeta}-z\omega^{2k})^2} + \frac{(\zeta^2-\omega^{-2k})^2}{(\omega^{-2k}-z\zeta)^2} + \frac{(\zeta\omega^{2k}-\zeta)^2}{(\zeta-z\omega^{2k})^2}\right]\Big\}.
\end{aligned} \tag{6.24}
$$

In addition,

$$
\partial_{\nu_z}\partial_z\partial_{\overline{z}}\widehat{h}_4(z,\zeta) = \begin{cases}
\partial_{\nu_z}N_3 + \frac{1}{2}|\zeta-z|^2(z\overline{\zeta}+\overline{z}\zeta-2)N_1 + 2n|\zeta-z|^4 \\
\quad -\frac{1}{6}|\zeta-z|^2\mathrm{Re}\Big\{2\overline{z}(\zeta-z)^2\partial_z N_1 - |\zeta-z|^2(z\zeta-z^2)\partial_z^2 N_1\Big\}, \\
\qquad\qquad z\in L\backslash\{1,\omega\}, \\
-\frac{i}{12}|\zeta-z|^2(\overline{\zeta}-\zeta)\Big\{6N_1 - 2(\overline{\zeta}+\zeta-2z)\partial_z N_1 \\
\qquad -|\zeta-z|^2\partial_z^2 N_1\Big\}, \quad z\in(0,1), \\
\frac{i}{12}|\zeta-z|^2(\overline{\zeta}e^{i\theta}-\zeta e^{-i\theta})\Big\{6N_1 - e^{2\theta i}|\zeta-z|^2\partial_z^2 N_1 \\
\qquad -2(\overline{\zeta}e^{2i\theta}+\zeta-2z)\partial_z N_1\Big\}, \quad z\in(\omega,0).
\end{cases} \tag{6.25}
$$

$$
\partial_{\nu_z}\widehat{h}_4(z,\zeta) = \begin{cases}
\partial_{\nu_z}N_4 + \frac{1}{12}|\zeta-z|^4(z\overline{\zeta}+\overline{z}\zeta-2)N_1(z,\zeta) + \frac{1}{9}n|\zeta-z|^6, \\
\qquad z\in L\backslash\{1,\omega\}, \\
-\frac{i}{12}|\zeta-z|^4(\overline{\zeta}-\zeta)N_1(z,\zeta), \quad z\in(0,1), \\
\frac{i}{12}|\zeta-z|^4(\overline{\zeta}e^{i\theta}-\zeta e^{-i\theta})N_1(z,\zeta), \quad z\in(\omega,0),
\end{cases} \tag{6.26}
$$

$$\partial_{\nu_z}(\partial_z\partial_{\overline{z}})^2\widehat{h}_4(z,\zeta)$$

$$= \begin{cases} \partial_{\nu_z}N_2 + (z\overline{\zeta}+\overline{z}\zeta-2)N_1 + 12n|\zeta-z|^2 + 2\mathrm{Re}\Big\{ -\overline{z}(\zeta-z)^2\partial_z N_1 \\ + \left[\frac{3}{2}z(\zeta-z)|\zeta-z|^2 + \frac{1}{6}\overline{z}(\zeta-z)^3\right]\partial_z^2 N_1 - \frac{1}{6}z|\zeta-z|^2(\zeta-z)^2\partial_z^3 N_1\Big\}, \\ \qquad z\in L\backslash\{1,\omega\}, \\ -i(\overline{\zeta}-\zeta)\Big\{N_1 - (\overline{\zeta}+\zeta-2z)\partial_z N_1 + \frac{1}{6}|\zeta-z|^2(\overline{\zeta}+\zeta-2z)\partial_z^3 N_1 \\ - \left[\frac{3}{2}|\zeta-z|^2 - \frac{1}{6}(\overline{\zeta}^2+|\zeta|^2+\zeta^2-3z\zeta-3z\overline{\zeta}+3z^2)\right]\partial_z^2 N_1\Big\}, \quad z\in(0,1), \\ i(\overline{\zeta}e^{i\theta}-\zeta e^{-i\theta})\Big\{N_1 - (\overline{\zeta}e^{2i\theta}+\zeta-2z)\partial_z N_1 + \frac{1}{6}|\zeta-z|^2 e^{2i\theta}(\overline{\zeta}e^{2i\theta}+\zeta-2z)\partial_z^3 N_1 \\ - \left[\frac{3}{2}e^{2i\theta}|\zeta-z|^2 - \frac{1}{6}(\overline{\zeta}^2e^{4i\theta}+|\zeta|^2e^{2i\theta}+\zeta^2-3z\zeta-3z\overline{\zeta}e^{2i\theta}+3z^2)\right]\partial_z^2 N_1\Big\}, \\ \qquad z\in(\omega,0). \end{cases} \tag{6.27}$$

Then a tetra-harmonic function is introduced by

$$\begin{aligned} F_5(z,\zeta) \;&= \frac{11}{27}n(|\zeta|^6+|z|^6) + \frac{1}{36}(\overline{z}\zeta+z\overline{\zeta})^3N_1 + \frac{1}{12}(\overline{z}\zeta+z\overline{\zeta})(|\zeta|^2+|z|^2)^2N_1 \\ &-\frac{1}{12}(\overline{z}^2\zeta^2+z^2\overline{\zeta}^2)(|z|^2+|\zeta|^2)N_1(z,\zeta) \\ &+\sum_{k=0}^{n-1}\Bigg\{\Bigg[\frac{(z\zeta\omega^{2k}+\overline{z}\overline{\zeta}\omega^{-2k})^3}{36|\zeta|^6} + \frac{(z\zeta\omega^{2k}+\overline{z}\overline{\zeta}\omega^{-2k})}{12|\zeta|^2}(|\zeta|^2+|z|^2)^2 \\ &\quad -\frac{(z^2\zeta^2\omega^{4k}+\overline{z}^2\overline{\zeta}^2\omega^{-4k})}{12|\zeta|^4}(|\zeta|^2+|z|^2)\Bigg]\log|\omega^{-2k}-z\zeta|^2 \\ &+\Bigg[\frac{(z\overline{\zeta}\omega^{2k}+\overline{z}\zeta\omega^{-2k})^3}{36|\zeta|^6} + \frac{(z\overline{\zeta}\omega^{2k}+\overline{z}\zeta\omega^{-2k})}{12|\zeta|^2}(|\zeta|^2+|z|^2)^2 \\ &\quad -\frac{(\overline{z}^2\zeta^2\omega^{-4k}+z^2\overline{\zeta}^2\omega^{4k})}{12|\zeta|^4}(|\zeta|^2+|z|^2)\Bigg]\log|\omega^{2k}-\overline{z}\zeta|^2 \\ &+\Bigg[\frac{1}{36}(z\overline{\zeta}\omega^{2k}+\overline{z}\zeta\omega^{-2k})^3 + \frac{1}{12}(z\overline{\zeta}\omega^{2k}+\overline{z}\zeta\omega^{-2k})(|\zeta|^2+|z|^2)^2 \\ &\quad -\frac{1}{12}(\overline{z}^2\zeta^2\omega^{-4k}+z^2\overline{\zeta}^2\omega^{4k})(|\zeta|^2+|z|^2)\Bigg]\log|\zeta-z\omega^{2k}|^2 \end{aligned}$$

$$+\left[\frac{1}{36}(z\zeta\omega^{2k}+\overline{z}\overline{\zeta}\omega^{-2k})^3+\frac{1}{12}(z\zeta\omega^{2k}+\overline{z}\overline{\zeta}\omega^{-2k})(|\zeta|^2+|z|^2)^2 \right.$$
$$\left. -\frac{1}{12}(\overline{z}^2\overline{\zeta}^2\omega^{-4k}+z^2\zeta^2\omega^{4k})(|\zeta|^2+|z|^2)\right]\log|\overline{\zeta}-z\omega^{2k}|^2\Bigg\}. \tag{6.28}$$

Let

$$\widetilde{F_5}(z,\zeta)=\widehat{h}_4(z,\zeta)-F_5(z,\zeta), \qquad z,\zeta\in\Omega, \tag{6.29}$$

By computation, we obtain that $\widetilde{F_5}(z,\zeta)$ satisfies,

$$\begin{cases} (\partial_z\partial_{\overline{z}})^3\widetilde{F_5}(z,\zeta)=0, & z,\ \zeta\in\Omega,\\ \partial_{\nu_z}\widetilde{F_5}(z,\zeta)=0, & z\in(\omega,0)\cup(0,1),\\ \partial_{\nu_z}\partial_z\partial_{\overline{z}}\widetilde{F_5}(z,\zeta)=0, & z\in(\omega,0)\cup(0,1),\\ \partial_{\nu_z}(\partial_z\partial_{\overline{z}})^2\widetilde{F_5}(z,\zeta)=0, & z\in(\omega,0)\cup(0,1). \end{cases} \tag{6.30}$$

Furthermore,

$$\widetilde{F_5}(z,\zeta)=-\frac{11}{27}(|\zeta|^6+1)n-\frac{1}{36}\left[(|\zeta|^3+1)^3+6|\zeta|^2(|\zeta|^2+1)\right]N_1(z,\zeta)$$
$$-2\sum_{k=0}^{n-1}\mathrm{Re}\{\Theta_1(z\omega^{2k},\zeta)+\Theta_1(\overline{z}\omega^{-2k},\zeta)\},\quad z\in L, \tag{6.31}$$

$$\partial_z\partial_{\overline{z}}\widetilde{F_5}(z,\zeta)=N_3-\frac{1}{4}\left(|\zeta|^4+4|\zeta|^2+1\right)N_1+\left(|\zeta|^4+2|\zeta|^2-\frac{10}{3}\right)n$$
$$-2\sum_{k=0}^{n-1}\mathrm{Re}\{\Theta_2(z\omega^{2k},\zeta)+\Theta_2(\overline{z}\omega^{-2k},\zeta)\},\quad z\in L, \tag{6.32}$$

$$(\partial_z\partial_{\overline{z}})^2\widetilde{F_5}(z,\zeta)=N_2-(|\zeta|^2+1)N_1+6n|\zeta|^2-\frac{34}{3}n$$
$$+2\sum_{k=0}^{n-1}\mathrm{Re}\{\Theta_3(z\omega^{2k},\zeta)+\Theta_3(\overline{z}\omega^{-2k},\zeta)\},\quad z\in L, \tag{6.33}$$

$$\partial_{\nu_z}\widetilde{F_5}(z,\zeta)=-\frac{1}{2}\left(|\zeta|^4+2|\zeta|^2+\frac{1}{3}\right)N_1+2n\left(|\zeta|^4-|\zeta|^2-\frac{1}{9}\right)$$
$$-2\sum_{k=0}^{n-1}\mathrm{Re}\{\Theta_4(z\omega^{2k},\zeta)+\Theta_4(\overline{z}\omega^{-2k},\zeta)\},\quad z\in L\backslash\{1,\omega\}, \tag{6.34}$$

$$\partial_{\nu_z}\partial_z\partial_{\overline{z}}\widetilde{F_5}(z,\zeta) = -\frac{46}{3}n + 12n|\zeta|^2 - (2|\zeta|^2+1)N_1 \\ -2\sum_{k=0}^{n-1}\mathrm{Re}\{\Theta_5(z\omega^{2k},\zeta)+\Theta_5(\overline{z}\omega^{-2k},\zeta)\}, \quad z\in L\backslash\{1,\omega\}, \tag{6.35}$$

$$\partial_{\nu_z}(\partial_z\partial_{\overline{z}})^2\widetilde{F_5}(z,\zeta) = -\frac{44}{3}n - 2N_1 + 2\sum_{k=0}^{n-1}\mathrm{Re}\{\Theta_6(z\omega^{2k},\zeta)+\Theta_6(\overline{z}\omega^{-2k},\zeta)\}, \\ z\in L\backslash\{1,\omega\}, \tag{6.36}$$

where

$$\Theta_1(z,\zeta) = \frac{1}{36}\left\{\frac{|\zeta|^6+1}{|\zeta|^6}(z^3\zeta^3+3z\zeta|\zeta|^2) + \frac{3(|\zeta|^2+1)^3}{|\zeta|^2}z\zeta \right. \\ \left. -\frac{3(|\zeta|^4+1)(|\zeta|^2+1)}{|\zeta|^4}z^2\zeta^2\right\}\log|1-z\zeta|^2, \tag{6.37}$$

$$\Theta_2(z,\zeta) = \frac{1}{12}\left\{\left[\frac{6|\zeta|^2+2}{|\zeta|^4}+6|\zeta|^2+10\right]z\zeta - \frac{3z^2(1+|\zeta|^4)}{\overline{\zeta}^2}\right\}\log|1-z\zeta|^2 \\ +\frac{1}{12(1-z\zeta)}\left\{-\frac{2}{|\zeta|^2}-|\zeta|^4-4|\zeta|^2-3+\frac{|\zeta|^4+1}{|\zeta|^4}\left[z^3\zeta^3-\overline{z}^2\overline{\zeta}^2\right]\right. \\ -z^2\zeta^2\left[\frac{2|\zeta|^2+1}{|\zeta|^4}+2|\zeta|^2+5\right]+z\zeta\left(|\zeta|^4+6|\zeta|^2+3\right) \\ \left.+\frac{\overline{z}(3|\zeta|^4+4|\zeta|^2+3)}{\zeta}\right\}, \tag{6.38}$$

$$\Theta_3(z,\zeta) = -\frac{z(|\zeta|^2+1)}{\overline{\zeta}}\log|1-z\zeta|^2 + \frac{1}{3(1-z\zeta)}\left\{5|\zeta|^2+7+\frac{1}{|\zeta|^2}\right. \\ \left.+\frac{3(1+|\zeta|^2)}{|\zeta|^2}z^2\zeta^2-\left(6+3|\zeta|^2\right)z\zeta-\left(\frac{3}{|\zeta|^2}+3\right)\overline{z}\overline{\zeta}\right\} \\ +\frac{1}{12(1-z\zeta)^2}\left\{-14|\zeta|^2-10+\frac{2}{|\zeta|^2}\left(z\zeta+\overline{z}\overline{\zeta}+z^3\zeta^3+z\zeta|\zeta|^2\right)\right. \\ \left.+10z\zeta(|\zeta|^2+1)+2(\overline{z}\overline{\zeta}+z^3\zeta^3)-6z^2\zeta^2-2z^2\zeta^2|\zeta|^2\right\}, \tag{6.39}$$

$$
\begin{aligned}
\Theta_4(z,\zeta) = \frac{1}{12}\Bigg\{ & \frac{|\zeta|^6+1}{|\zeta|^6}(z^3\zeta^3+3z\zeta|\zeta|^2) + \frac{(|\zeta|^2+1)^2}{|\zeta|^2}(|\zeta|^2+5)z\zeta \\
& - \frac{2(|\zeta|^4+1)(|\zeta|^2+2)}{|\zeta|^4}z^2\zeta^2\Bigg\} \log|1-z\zeta|^2 \\
+ \frac{1}{36(1-z\zeta)}\Bigg\{ & \frac{|\zeta|^6-1}{|\zeta|^6}(z\zeta+\overline{z}\overline{\zeta})^3 + \frac{3(|\zeta|^4-1)(|\zeta|^2+1)}{|\zeta|^2}(z\zeta+\overline{z}\overline{\zeta}) \\
& - \frac{3(|\zeta|^4-1)(|\zeta|^2+1)}{|\zeta|^4}(z^2\zeta^2+\overline{z}^2\overline{\zeta}^2)\Bigg\} \\
& + \frac{1}{18|\zeta|^6}(z^3\zeta^3+3z\zeta|\zeta|^2) + \frac{(|\zeta|^2+1)^2}{6|\zeta|^2}z\zeta - \frac{|\zeta|^2+1}{6|\zeta|^4}z^2\zeta^2,
\end{aligned} \tag{6.40}
$$

$$
\begin{aligned}
\Theta_5(z,\zeta) = \frac{1}{12}\Bigg\{ & \left(\frac{18|\zeta|^2+2}{|\zeta|^4}+6|\zeta|^2+22\right)z\zeta - \frac{6(1+|\zeta|^4)}{|\zeta|^4}z^2\zeta^2\Bigg\} \log|1-z\zeta|^2 \\
& + \frac{1}{12(1-z\zeta)^2}\Bigg\{\overline{z}^2\overline{\zeta}^2 - \frac{1+3|\zeta|^4+2|\zeta|^2}{|\zeta|^2}\overline{z}\overline{\zeta} - \frac{4}{|\zeta|^2}+4|\zeta|^4+6|\zeta|^2-1\Bigg\} \\
& + \frac{1}{12(1-z\zeta)}\Bigg\{-\frac{\overline{z}^2(6|\zeta|^4+2)}{\zeta^2} + \left(12|\zeta|^2+\frac{12}{|\zeta|^2}+16\right)\overline{z}\overline{\zeta} \\
& \qquad + \frac{1}{|\zeta|^4} - 4|\zeta|^4 - \frac{12}{|\zeta|^2} - 2|\zeta|^2 - 11\Bigg\} \\
& + \frac{1}{12}\Bigg\{-\frac{z^2(5+2|\zeta|^4)}{\overline{\zeta}^2} + z\zeta\left[\frac{12}{|\zeta|^2}-\frac{1}{|\zeta|^4}+2|\zeta|^2+15\right] \\
& \qquad - \frac{1}{|\zeta|^4} - 14|\zeta|^2 + \frac{10}{|\zeta|^2} - 3\Bigg\},
\end{aligned} \tag{6.41}
$$

and

$$
\begin{aligned}
\Theta_6(z,\zeta) = & -\frac{z(|\zeta|^2+1)}{\overline{\zeta}}\log|1-z\zeta|^2 \\
& + \frac{1}{1-z\zeta}\Bigg\{3|\zeta|^2+\frac{22}{3}+\frac{3z^2\zeta^2}{|\zeta|^2}-5z\zeta+z^2\zeta^2-\left(\frac{1}{|\zeta|^2}+3\right)\overline{z}\overline{\zeta}\Bigg\} \\
& + \frac{1}{(1-z\zeta)^2}\Bigg\{-3|\zeta|^2-\frac{13}{3}+\left(\frac{3}{2|\zeta|^2}+\frac{1}{6}\right)z^3\zeta^3+\left(7+\frac{5}{2}|\zeta|^2+\frac{1}{2|\zeta|^2}\right)z\zeta \\
& \qquad -2z^2\zeta^2+\left(\frac{3}{2}+\frac{1}{6|\zeta|^2}\right)\overline{z}\overline{\zeta}\Bigg\}
\end{aligned}
$$

$$
+\frac{1}{3(1-z\zeta)^3}\left\{1+\left(\frac{1}{|\zeta|^2}+6+3|\zeta|^2\right)z^2\zeta^2-\left(6+4|\zeta|^2\right)z\zeta
+\frac{z^4\zeta^4}{|\zeta|^2}-z^3\zeta^3-\overline{z}\overline{\zeta}\right\}. \tag{6.42}
$$

Therefore, the following theorem is true.

**Theorem 6.2.1.** *For* $z,\zeta\in\Omega$,

$$
\begin{aligned}
\widehat{h}_4(z,\zeta) &= F_5(z,\zeta)+b_0+(|z|^2-1)b_1+\left[\frac{1}{4}(|z|^2-1)^2-\frac{1}{2}(|z|^2-1)\right]b_2\\
&-\frac{1}{2}\left(|\zeta|^4+2|\zeta|^2+\frac{1}{3}\right)A_0-(2|\zeta|^2+1)A_1-2A_2\\
&-\mathrm{Re}\left\{\frac{1}{2\pi i}\int\limits_{|\widetilde{\zeta}|=1}\left[N_1(z,\widetilde{\zeta})\Theta_4(\widetilde{\zeta},\zeta)+N_2(z,\widetilde{\zeta})\Theta_5(\widetilde{\zeta},\zeta)\right.\right.\\
&\left.\left.-N_3(z,\widetilde{\zeta})\Theta_6(\widetilde{\zeta},\zeta)\right]\frac{\mathrm{d}\widetilde{\zeta}}{\widetilde{\zeta}}\right\},
\end{aligned}
$$

*where* $F_5$, $\Theta_i$ $(i=4,5,6)$ *are defined by* (6.28), (6.40)-(6.42) *and*

$$
\begin{cases}
b_0=\left(-\frac{11}{54}|\zeta|^6+\frac{7}{6}|\zeta|^4+\frac{1}{3|\zeta|^2}+\frac{5}{6}|\zeta|^2-\frac{11}{54}\right)n,\\
b_1=\left(\frac{13}{6}|\zeta|^4+\frac{4}{3|\zeta|^2}+\frac{16}{3}|\zeta|^2-\frac{3}{2}\right)n,\\
b_2=\frac{n}{3}\left(28|\zeta|^2+\frac{4}{|\zeta|^2}-12\right).
\end{cases} \tag{6.43}
$$

*Besides,*

$$
\begin{aligned}
A_1 &= -8n(|z|^2+1)+\sum_{k=0}^{n-1}\sum_{l=0}^{\infty}\left[\frac{2(|z|^2+1)}{(l+1)^2}-\frac{4}{(l+1)^3}\right]\\
&\times\left[(z\overline{\zeta}\omega^{2k})^{l+1}+(\overline{z}\zeta\omega^{-2k})^{l+1}+(z\zeta\omega^{2k})^{l+1}+(\overline{z}\overline{\zeta}\omega^{-2k})^{l+1}\right]\\
&+2(|z|^2+1)\sum_{k=0}^{n-1}\left\{\log\left|(1-z\zeta\omega^{2k})(1-z\overline{\zeta}\omega^{2k})\right|^2-\frac{\log(1-z\zeta\omega^{2k})}{z\zeta\omega^{2k}}\right.\\
&\left.-\frac{\log(1-\overline{z}\zeta\omega^{-2k})}{\overline{z}\zeta\omega^{-2k}}-\frac{\log(1-z\overline{\zeta}\omega^{2k})}{z\overline{\zeta}\omega^{2k}}-\frac{\log(1-\overline{z}\overline{\zeta}\omega^{-2k})}{\overline{z}\overline{\zeta}\omega^{-2k}}\right\},
\end{aligned} \tag{6.44}
$$

$$A_0 = \sum_{k=0}^{n-1}\sum_{l=0}^{\infty}\frac{2}{(l+1)^2}\Big[(z\overline{\zeta}\omega^{2k})^{l+1} + (\overline{z}\zeta\omega^{-2k})^{l+1} + (z\zeta\omega^{2k})^{l+1} + (\overline{z}\overline{\zeta}\omega^{-2k})^{l+1}\Big], \tag{6.45}$$

$$\begin{aligned} A_2 \; &= \sum_{k=0}^{n-1}\sum_{l=0}^{\infty}\left[\frac{8}{(l+1)^4} - \frac{10+4|z|^2}{(l+1)^3} - \frac{2(|z|^2+1)}{(l+2)^2(l+1)} + \frac{12|z|^2+|z|^4+25}{2(l+1)^2}\right] \\ &\quad\times\Big[(z\overline{\zeta}\omega^{2k})^{l+1} + (\overline{z}\zeta\omega^{-2k})^{l+1} + (z\zeta\omega^{2k})^{l+1} + (\overline{z}\overline{\zeta}\omega^{-2k})^{l+1}\Big] \\ &+ \left[|z|^4 + 6|z|^2 + 13 - \frac{1}{4}(|z|^4+1)\right]\sum_{k=0}^{n-1}\log\left|(1-z\zeta\omega^{2k})(1-z\overline{\zeta}\omega^{2k})\right|^2 \\ &- [|z|^4 + 6|z|^2 + 13]\sum_{k=0}^{n-1}\left\{\frac{\log(1-z\zeta\omega^{2k})}{z\zeta\omega^{2k}} + \frac{\log(1-\overline{z}\zeta\omega^{-2k})}{\overline{z}\zeta\omega^{-2k}}\right. \\ &\quad \left. + \frac{\log(1-z\overline{\zeta}\omega^{2k})}{z\overline{\zeta}\omega^{2k}} + \frac{\log(1-\overline{z}\overline{\zeta}\omega^{-2k})}{\overline{z}\overline{\zeta}\omega^{-2k}} + 4\right\} \\ &+ \frac{1}{4}(|z|^4+1)\sum_{k=0}^{n-1}\left\{\frac{\log(1-z\zeta\omega^{2k})}{(z\zeta\omega^{2k})^2} + \frac{\log(1-\overline{z}\zeta\omega^{-2k})}{(\overline{z}\zeta\omega^{-2k})^2}\right. \\ &+ \left.\frac{\log(1-z\overline{\zeta}\omega^{2k})}{(z\overline{\zeta}\omega^{2k})^2} + \frac{\log(1-\overline{z}\overline{\zeta}\omega^{-2k})}{(\overline{z}\overline{\zeta}\omega^{-2k})^2} + \frac{(z+\overline{z})(\zeta+\overline{\zeta})}{|\zeta|^2}\omega^{2k} + 2\right\}. \end{aligned} \tag{6.46}$$

**Proof:** We firstly prove that the first solvability condition for $\widetilde{F_5}(z,\zeta)$ in Theorem 5.3.4 is true, that is,

$$\frac{1}{2\pi}\int_{\partial\Omega}\partial_{\nu_z}(\partial_z\partial_{\overline{z}})^2\widetilde{F_5}(z,\zeta)\mathrm{d}s_z = \frac{2}{\pi}\int_{\Omega}(\partial_z\partial_{\overline{z}})^3\widetilde{F_5}(z,\zeta)\mathrm{d}x\mathrm{d}y, \tag{6.47}$$

with $z = x+iy,\ x,\ y \in \mathbb{R}$. By (6.30) and (6.36), we know that the right-hand side in (6.47) equals 0 and the left-hand side is

$$\frac{1}{2\pi i}\int_{L}\partial_{\nu_z}(\partial_z\partial_{\overline{z}})^2\widetilde{F_5}(z,\zeta)\frac{\mathrm{d}z}{z} = -\frac{22}{3}n + 2\mathrm{Re}\left\{\frac{1}{2\pi i}\int_{|z|=1}\Theta_6(z,\zeta)\frac{\mathrm{d}z}{z}\right\} = 0.$$

Thus, the first condition (6.47) is valid.

Next, the second solvability condition is

$$
\begin{aligned}
&\frac{n}{\pi}\int\limits_{\partial\Omega}\partial_{\nu_z}\partial_z\partial_{\overline{z}}\widetilde{F_5}(z,\zeta)\mathrm{d}s_z+\frac{n}{\pi}\int\limits_{\partial\Omega}(|z|^2-1)\partial_{\nu_z}(\partial_z\partial_{\overline{z}})^2\widetilde{F_5}(z,\zeta)\mathrm{d}s_z\\
&=\frac{2n}{\pi i}\int\limits_{L}(\partial_z\partial_{\overline{z}})^2\widetilde{F_5}(z,\zeta)\frac{\mathrm{d}z}{z}+\frac{4n}{\pi}\int\limits_{\Omega}(|z|^2-1)(\partial_z\partial_{\overline{z}})^3\widetilde{F_5}(z,\zeta)\mathrm{d}x\mathrm{d}y,
\end{aligned}
\tag{6.48}
$$

then from (6.30), the second term of the left-hand side and the right-hand side in (6.48) vanish respectively. Moreover by (6.33),

$$
\begin{aligned}
\frac{n}{\pi i}\int\limits_{L}(\partial_z\partial_{\overline{z}})^2\widetilde{F_5}(z,\zeta)\frac{\mathrm{d}z}{z}&=\left(6|\zeta|^2-\frac{34}{3}\right)n+4n\mathrm{Re}\left\{\frac{1}{2\pi i}\int\limits_{|z|=1}\Theta_3(z,\zeta)\frac{\mathrm{d}z}{z}\right\}\\
&=\frac{n}{3}\left(28|\zeta|^2+\frac{4}{|\zeta|^2}-12\right).
\end{aligned}
\tag{6.49}
$$

Besides, from (6.30) and (6.35),

$$
\begin{aligned}
\frac{n}{\pi}\int\limits_{\partial\Omega}\partial_{\nu_z}\partial_z\partial_{\overline{z}}\widetilde{F_5}(z,\zeta)\mathrm{d}s_z&=\left(12|\zeta|^2-\frac{46}{3}\right)n-4n\mathrm{Re}\left\{\frac{1}{2\pi i}\int\limits_{|z|=1}\Theta_5(z,\zeta)\frac{\mathrm{d}z}{z}\right\}\\
&=\frac{2n}{3}\left(28|\zeta|^2+\frac{4}{|\zeta|^2}-12\right),
\end{aligned}
$$

which means that the second solvability condition (6.48) holds. Finally, we also need to verify the third solvability condition,

$$
\begin{aligned}
&\frac{n}{\pi}\int\limits_{\partial\Omega}\partial_{\nu_z}\widetilde{F_5}(z,\zeta)\mathrm{d}s_z+\frac{n}{\pi}\int\limits_{\partial\Omega}(|z|^2-1)\Bigg[\partial_{\nu_z}\partial_z\partial_{\overline{z}}\widetilde{F_5}(z,\zeta)\\
&\qquad+\frac{1}{4}(|z|^2-3)\partial_{\nu_z}(\partial_z\partial_{\overline{z}})^2\widetilde{F_5}(z,\zeta)\Bigg]\mathrm{d}s_z\\
&=\frac{2n}{\pi i}\int\limits_{L}\partial_z\partial_{\overline{z}}\widetilde{F_5}(z,\zeta)\frac{\mathrm{d}z}{z}-\frac{n}{\pi i}\int\limits_{L}(\partial_z\partial_{\overline{z}})^2\widetilde{F_5}(z,\zeta)\frac{\mathrm{d}z}{z}\\
&\qquad+\frac{n}{\pi}\int\limits_{\Omega}(|z|^2-1)(|z|^2-3)(\partial_z\partial_{\overline{z}})^3\widetilde{F_5}(z,\zeta)\mathrm{d}x\mathrm{d}y,
\end{aligned}
\tag{6.50}
$$

where the second term of the left-hand side and the third term of the right-hand side disappear and by (6.32),

$$\begin{aligned}&\frac{n}{\pi i}\int\limits_L \partial_z\partial_{\overline{z}}\widetilde{F_5}(z,\zeta)\frac{\mathrm{d}z}{z}\\&=\frac{n}{3}\left(3|\zeta|^4+6|\zeta|^2-10\right)-4n\mathrm{Re}\left\{\frac{1}{2\pi i}\int\limits_{|z|=1}\Theta_2(z,\zeta)\frac{\mathrm{d}z}{z}\right\}\\&=\left(\frac{13}{6}|\zeta|^4+\frac{4}{3|\zeta|^2}+\frac{16}{3}|\zeta|^2-\frac{3}{2}\right)n.\end{aligned}\tag{6.51}$$

From (6.34),

$$\begin{aligned}\frac{n}{\pi}\int\limits_{\partial\Omega}\partial_{\nu_z}\widetilde{F_5}(z,\zeta)\mathrm{d}s_z&=2n\left(|\zeta|^4-|\zeta|^2-\frac{1}{9}\right)-4n\mathrm{Re}\left\{\frac{1}{2\pi i}\int\limits_{|z|=1}\Theta_4(z,\zeta)\frac{\mathrm{d}z}{z}\right\}\\&=\left(\frac{13}{3}|\zeta|^4+\frac{4}{3|\zeta|^2}+\frac{4}{3}|\zeta|^2+1\right)n,\end{aligned}\tag{6.52}$$

thus, by (6.49), (6.51) and (6.52), we see that (6.50) holds. Moreover, from (6.31),

$$\begin{aligned}\frac{n}{\pi i}\int\limits_L\widetilde{F_5}(z,\zeta)\frac{\mathrm{d}z}{z}&=-\frac{11}{27}(|\zeta|^6+1)n-4n\mathrm{Re}\left\{\frac{1}{2\pi i}\int\limits_{|z|=1}\Theta_1(z,\zeta)\frac{\mathrm{d}z}{z}\right\}\\&=\left(-\frac{11}{54}|\zeta|^6+\frac{7}{6}|\zeta|^4+\frac{1}{3|\zeta|^2}+\frac{5}{6}|\zeta|^2-\frac{11}{54}\right)n.\end{aligned}\tag{6.53}$$

Hence, from Theorem 5.3.4,

$$\begin{aligned}\widetilde{F_5}(z,\zeta)&=b_0+(|z|^2-1)b_1+\left[\frac{1}{4}(|z|^2-1)^2-\frac{1}{2}(|z|^2-1)\right]b_2\\&\quad+\frac{1}{4\pi i}\int\limits_L\Big[N_1(z,\widetilde{\zeta})\partial_{\nu_{\widetilde{\zeta}}}\widetilde{F_5}(\widetilde{\zeta},\zeta)+N_2(z,\widetilde{\zeta})\partial_{\nu_{\widetilde{\zeta}}}\partial_{\widetilde{\zeta}\overline{\widetilde{\zeta}}}\widetilde{F_5}(\widetilde{\zeta},\zeta)\\&\qquad+N_3(z,\widetilde{\zeta})\partial_{\nu_{\widetilde{\zeta}}}(\partial_{\widetilde{\zeta}\overline{\widetilde{\zeta}}})^2\widetilde{F_5}(\widetilde{\zeta},\zeta)\Big]\frac{\mathrm{d}\widetilde{\zeta}}{\widetilde{\zeta}}.\end{aligned}\tag{6.54}$$

Actually, we also have

$$
\begin{cases}
A_0 = \dfrac{1}{4\pi i}\displaystyle\int\limits_L N_1(\widetilde{\zeta},\zeta)N_1(z,\widetilde{\zeta})\dfrac{d\widetilde{\zeta}}{\widetilde{\zeta}},\\
A_1 = \dfrac{1}{4\pi i}\displaystyle\int\limits_L N_1(\widetilde{\zeta},\zeta)N_2(z,\widetilde{\zeta})\dfrac{d\widetilde{\zeta}}{\widetilde{\zeta}},\\
A_2 = \dfrac{1}{4\pi i}\displaystyle\int\limits_L N_1(\widetilde{\zeta},\zeta)N_3(z,\widetilde{\zeta})\dfrac{d\widetilde{\zeta}}{\widetilde{\zeta}}.
\end{cases}
\tag{6.55}
$$

Therefore, from (6.34)-(6.36), (6.54) and (6.55), we obtain the desired conclusion.

Hence, by (6.20) and Theorem 6.2.1,

$$
\begin{aligned}
N_4(z,\zeta) \;&= \frac{1}{36}|\zeta - z|^6 N_1(z,\zeta) + F_5(z,\zeta) + b_0 + (|z|^2-1)b_1\\
&\quad + \left[\frac{1}{4}(|z|^2-1)^2 - \frac{1}{2}(|z|^2-1)\right] b_2\\
&\quad -\frac{1}{2}\left(|\zeta|^4 + 2|\zeta|^2 + \frac{1}{3}\right) A_0 - (2|\zeta|^2+1)A_1 - 2A_2\\
&\quad -\mathrm{Re}\Bigg\{\frac{1}{2\pi i}\int\limits_{|\widetilde{\zeta}|=1}\Bigg[N_1(z,\widetilde{\zeta})\Theta_4(\widetilde{\zeta},\zeta) + N_2(z,\widetilde{\zeta})\Theta_5(\widetilde{\zeta},\zeta)\\
&\qquad\qquad -N_3(z,\widetilde{\zeta})\Theta_6(\widetilde{\zeta},\zeta)\Bigg]\frac{d\widetilde{\zeta}}{\widetilde{\zeta}}\Bigg\},
\end{aligned}
$$

where $N_1$, $F_5$, $b_i$ $(i = 0,1,2)$, $A_i$ $(i = 0,1,2)$, $\Theta_i$ $(i = 3,4,5)$ are defined by (4.17), (6.28), (6.43)-(6.46) and (6.40)-(6.42) respectively.

**Remark 6.2.1.** From the procedure of constructing the m-harmonic Green and m-harmonic Neumann functions (m=2, 3, 4) explicitly for the fan-shaped domain $\Omega$ by convolution, we observe that the critical step is to construct a proper function, which establishes a bridge such that finally we only need to compute the boundary integrals on the whole unit circle. However as we see, the process is very complicated. Unfortunately, up to now, it is still not easy to construct the expressions of the polyharmonic Green and polyharmonic Neumann functions of arbitrary order $m$ explicitly.

# Chapter 7

# Polyharmonic Dirichlet and Polyharmonic Neumann Problems

As we know, by convolution, the expressions of iterated polyharmonic Green and polyharmonic Neumann functions can not be easily constructed in explicit form because of complicated computation. However, similarly to the unit disc, they can be defined recursively and used to solve the related higher order Dirichlet and Neumann problems for the m-Poisson equation in the fan-shaped domain $\Omega$. Also $\Omega,\ L,\ \omega,\ \theta$ are determined as before.

## 7.1 Polyharmonic Dirichlet Problem

The polyharmonic Green function of order $m$ is introduced by,

$$\widehat{G}_m(z,\zeta) = -\frac{1}{\pi}\int_{\Omega} G_1(z,\widetilde{\zeta})\widehat{G}_{m-1}(\widetilde{\zeta},\zeta)\mathrm{d}\widetilde{\xi}\mathrm{d}\widetilde{\eta}, \quad z,\zeta\in\Omega, \quad m\geq 2, \tag{7.1}$$

where $G_1(z,\zeta) = \widehat{G}_1(z,\zeta)$ is defined by (4.1). Then $\widehat{G}_m(z,\zeta)$ has the properties

1. $\widehat{G}_m(.,\zeta)$ is polyharmonic of order $m$ in $\Omega\backslash\{\zeta\}$,
2. $\widehat{G}_m(z,\zeta) + \dfrac{|\zeta - z|^{2(m-1)}}{(m-1)!^2}\log|\zeta - z|^2$ is polyharmonic of order $m$ for $z\in\Omega$,
3. $(\partial_z\partial_{\overline{z}})^{\mu}\widehat{G}_m(z,\zeta) = 0$ for $0\leq\mu\leq m-1$, $z\in\partial\Omega$, $\zeta\in\Omega$.
4. $\widehat{G}_m(z,\zeta) = \widehat{G}_m(\zeta,z)$ for $z,\zeta\in\Omega$.

Moreover, the following representation formula and the statement about the Dirichlet problem are valid [8, 27].

**Theorem 7.1.1.** *Any* $w\in C^{2m}(\Omega;\mathbb{C})\cap C^{2m-1}(\overline{\Omega};\mathbb{C}),\ m\in\mathbb{N},$ *can be expressed* *as*

$$w(z) = -\sum_{\mu=1}^{m} \frac{1}{4\pi} \int_{\partial\Omega} \partial_{\nu_\zeta} \widehat{G}_\mu(z,\zeta)(\partial_\zeta\partial_{\overline{\zeta}})^{\mu-1} w(\zeta) \mathrm{d}s_\zeta - \frac{1}{\pi} \int_{\Omega} \widehat{G}_m(z,\zeta)(\partial_\zeta\partial_{\overline{\zeta}})^m w(\zeta) \mathrm{d}\xi \mathrm{d}\eta,$$

*where* $\widehat{G}_\mu$ $(\mu = 1, \cdots, m)$ *are the polyharmonic Green functions of order* $\mu$ *respectively.*

**Theorem 7.1.2.** *The Dirichlet problem*

$$(\partial_\zeta\partial_{\overline{\zeta}})^m w = f \quad in \quad \Omega, \quad (\partial_\zeta\partial_{\overline{\zeta}})^\mu w = \gamma_\mu, \quad 0 \le \mu \le m-1 \quad on \quad \partial\Omega,$$

$f \in L_p(\Omega;\mathbb{C}),\ p > 2,\ \gamma_\mu \in C(\partial\Omega;\mathbb{C}),\ 0 \le \mu \le m-1,$ *is uniquely solvable by*

$$w(z) = -\sum_{\mu=1}^{m} \frac{1}{4\pi} \int_{\partial\Omega} \partial_{\nu_\zeta} \widehat{G}_\mu(z,\zeta)\gamma_{\mu-1}(\zeta) \mathrm{d}s_\zeta - \frac{1}{\pi} \int_{\Omega} \widehat{G}_m(z,\zeta) f(\zeta) \mathrm{d}\xi \mathrm{d}\eta,$$

$$z \in \Omega,$$

*where* $\widehat{G}_1 = G_1$, $\widehat{G}_\mu(z,\zeta)$ $(m \ge 2)$ *are defined in* (4.1) *and* (7.1) *respectively.*

## 7.2 Polyharmonic Neumann Problem

Similarly, convoluting the harmonic Neumann function $N_1(z,\zeta)$ with $N_{m-1}(z,\zeta)$ leads to a higher order Neumann function

$$N_m(z,\zeta) = -\frac{1}{\pi} \int_{\Omega} N_1(z,\widetilde{\zeta}) N_{m-1}(\widetilde{\zeta},\zeta) \mathrm{d}\widetilde{\xi} \mathrm{d}\widetilde{\eta}, \quad z,\zeta \in \Omega, \quad m \ge 2, \tag{7.2}$$

with $N_1(z,\zeta)$ given by (4.17). And obviously it has the properties

$$\partial_z \partial_{\overline{z}} N_m(z,\zeta) = N_{m-1}(z,\zeta), \quad z,\ \zeta \in \Omega, \tag{7.3}$$

$$\partial_{\nu_z} N_m(z,\zeta) = \begin{cases} \dfrac{4n}{\pi} \displaystyle\int_{\Omega} N_{m-1}(\widetilde{\zeta},\zeta) \mathrm{d}\widetilde{\xi} \mathrm{d}\widetilde{\eta}, & z \in L \backslash \{1,\omega\}, \\ 0, & z \in (\omega,0) \cup (0,1), \end{cases} \quad m \ge 2. \tag{7.4}$$

Then (7.4) implies that $\partial_{\nu_z} N_m(z,\zeta)$ is independent of $z$ for $z \in \partial\Omega$. Moreover, the normalization condition is also true,

$$\int_{L} N_m(z,\zeta) \frac{\mathrm{d}z}{z} = 0, \qquad m \ge 1. \tag{7.5}$$

Next, we give the following higher order representation formula.

**Theorem 7.2.1.** *Any* $w \in C^{2m}(\Omega;\mathbb{C}) \cap C^{2m-1}(\overline{\Omega};\mathbb{C})$, $m \in \mathbb{N}$, *can be expressed as*

$$w(z) = \sum_{\mu=0}^{m-1} \frac{1}{4\pi} \int_{\partial\Omega} \left[ N_{\mu+1}(z,\zeta) \partial_{\nu_\zeta} (\partial_\zeta \partial_{\overline{\zeta}})^\mu w(\zeta) - \partial_{\nu_\zeta} N_{\mu+1}(z,\zeta) (\partial_\zeta \partial_{\overline{\zeta}})^\mu w(\zeta) \right] ds_\zeta$$
$$- \frac{1}{\pi} \int_\Omega N_m(z,\zeta) (\partial_\zeta \partial_{\overline{\zeta}})^m w(\zeta) d\xi d\eta, \quad z \in \Omega. \tag{7.6}$$

**Proof:** By Theorem 3.2.1 and Theorem 4.3.3, (7.6) is true for $m = 1,\ 2$. Suppose that (7.6) is valid for $m \le k-1$ $(k \ge 2)$, then when $w \in C^{2k}(\Omega;\mathbb{C}) \cap C^{2k-1}(\overline{\Omega};\mathbb{C})$, the above representation formula of $m = k-1$ is true for $\partial_z \partial_{\overline{z}} w(z)$, that is,

$$\partial_z \partial_{\overline{z}} w(z) = \sum_{\mu=0}^{k-2} \frac{1}{4\pi} \int_{\partial\Omega} \Bigg[ N_{\mu+1}(z,\zeta) \partial_{\nu_\zeta} (\partial_\zeta \partial_{\overline{\zeta}})^{\mu+1} w(\zeta)$$
$$- \partial_{\nu_\zeta} N_{\mu+1}(z,\zeta) (\partial_\zeta \partial_{\overline{\zeta}})^{\mu+1} w(\zeta) \Bigg] ds_\zeta \tag{7.7}$$
$$- \frac{1}{\pi} \int_\Omega N_{k-1}(z,\zeta) (\partial_\zeta \partial_{\overline{\zeta}})^k w(\zeta) d\xi d\eta.$$

In addition, we have

$$w(z) = \frac{1}{4\pi} \int_{\partial\Omega} \left[ \partial_{\nu_\zeta} w(\zeta) N_1(z,\zeta) - w(\zeta) \partial_{\nu_\zeta} N_1(z,\zeta) \right] ds_\zeta$$
$$- \frac{1}{\pi} \int_\Omega N_1(z,\zeta) \partial_\zeta \partial_{\overline{\zeta}} w(\zeta) d\xi d\eta. \tag{7.8}$$

Inserting (7.7) into the area integral of (7.8) leads to

$$- \frac{1}{\pi} \int_\Omega N_1(z,\zeta) \partial_\zeta \partial_{\overline{\zeta}} w(\zeta) d\xi d\eta$$
$$= \sum_{\mu=0}^{k-2} \frac{1}{4\pi} \int_{\partial\Omega} \Bigg\{ \partial_{\nu_{\widetilde{\zeta}}} (\partial_{\widetilde{\zeta}} \partial_{\overline{\widetilde{\zeta}}})^{\mu+1} w(\widetilde{\zeta}) \left[ - \frac{1}{\pi} \int_\Omega N_1(z,\zeta) N_{\mu+1}(\zeta,\widetilde{\zeta}) d\xi d\eta \right]$$
$$- (\partial_{\widetilde{\zeta}} \partial_{\overline{\widetilde{\zeta}}})^{\mu+1} w(\widetilde{\zeta}) \left[ - \frac{1}{\pi} \int_\Omega N_1(z,\zeta) \partial_{\nu_{\widetilde{\zeta}}} N_{\mu+1}(\zeta,\widetilde{\zeta}) d\xi d\eta \right] \Bigg\} ds_{\widetilde{\zeta}} \tag{7.9}$$
$$+ \frac{1}{\pi} \int_\Omega (\partial_{\widetilde{\zeta}} \partial_{\overline{\widetilde{\zeta}}})^k w(\widetilde{\zeta}) \left[ \frac{1}{\pi} \int_\Omega N_1(z,\zeta) N_{k-1}(\zeta,\widetilde{\zeta}) d\xi d\eta \right] d\widetilde{\xi} d\widetilde{\eta},$$

then, we see that (7.6) is also valid for $m = k$ due to (7.2) and (7.9). Thus, the proof is completed.

Next, we restate several lemmas.

**Lemma 7.2.1.** [23] *For any $m,\ k \in \mathbb{N}$,*

$$(\partial_z\partial_{\overline{z}})^{\sigma}\frac{1}{k!^2}(|z|^2-1)^k = \sum_{\tau=0}^{\sigma}\left(\begin{array}{c}\sigma\\ \tau\end{array}\right)\frac{(k-\tau)!}{k!(k-\sigma)!(k-\sigma-\tau)!}(|z|^2-1)^{k-\sigma-\tau},\quad 0\le 2\sigma\le k,$$

$$\begin{aligned}&(\partial_z\partial_{\overline{z}})^{m+k}\frac{1}{(2m-1)!^2}(|z|^2-1)^{2m-1}\\ &=\sum_{\tau=0}^{m-k-1}\left(\begin{array}{c}m+k\\ \tau\end{array}\right)\frac{(2m-\tau-1)!(|z|^2-1)^{m-1-k-\tau}}{(2m-1)!(m-1-k)!(m-1-k-\tau)!},\quad 0\le k\le m-1,\end{aligned}$$

$$(\partial_z\partial_{\overline{z}})^{m+k}\frac{1}{(2m)!^2}(|z|^2-1)^{2m} = \sum_{\tau=0}^{m-k}\left(\begin{array}{c}m+k\\ \tau\end{array}\right)\frac{(2m-\tau)!(|z|^2-1)^{m-k-\tau}}{(2m)!(m-k)!(m-k-\tau)!},\quad 0\le k\le m.$$

**Lemma 7.2.2.** [23] *For $|z| = 1$,*

$$(\partial_z\partial_{\overline{z}})^{\sigma}\frac{1}{(2m-1)!^2}(|z|^2-1)^{2m-1} = 0,\qquad 0\le\sigma\le m-1,$$

$$(\partial_z\partial_{\overline{z}})^{m+k}\frac{1}{(2m-1)!^2}(|z|^2-1)^{2m-1} = \frac{(m+k)!^2}{(2m-1)!(m-1-k)!^2(2k+1)!},\quad 0\le k\le m-1,$$

$$(\partial_z\partial_{\overline{z}})^{\sigma}\frac{1}{(2m)!^2}(|z|^2-1)^{2m} = 0,\qquad 0\le\sigma\le m-1,$$

$$(\partial_z\partial_{\overline{z}})^{m+k}\frac{1}{(2m)!^2}(|z|^2-1)^{2m} = \frac{(m+k)!^2}{(2m)!(m-k)!^2(2k)!},\quad 0\le k\le m.$$

**Remark 7.2.1.** Obviously, we see that for $m \in \mathbb{N}$,

$$\left\{\begin{array}{l}(\partial_z\partial_{\overline{z}})^{m}\frac{1}{m!^2}(|z|^2-1)^m = 1,\quad z\in\Omega,\\ \partial_{\nu_z}(\partial_z\partial_{\overline{z}})^{\sigma}\frac{1}{m!^2}(|z|^2-1)^m = 0,\ \ 0\le\sigma\le m-1,\quad z\in(\omega,0)\cup(0,1).\end{array}\right. \tag{7.10}$$

Then, the following result is valid.

**Lemma 7.2.3.** *For any* $k,\ m \in \mathbb{N},\ 2k-1 \le m-1$ *and* $2k \le m-1$,

$$-\frac{1}{\pi}\int\limits_{\Omega} N_{2k-1}(z,\zeta)\mathrm{d}x\mathrm{d}y = \frac{1}{(2k-1)!^2}(|\zeta|^2-1)^{2k-1} + \frac{1}{4n}\sum_{\mu=k}^{2(k-1)} \frac{\mu!^2}{(2k-1)!(2k-1-\mu)!^2(2\mu-2k+1)!}\partial_{\nu_z}N_{\mu+1}(z,\zeta), \tag{7.11}$$

$$-\frac{1}{\pi}\int\limits_{\Omega} N_{2k}(z,\zeta)\mathrm{d}x\mathrm{d}y = \frac{1}{(2k)!^2}(|\zeta|^2-1)^{2k} + \frac{1}{4n}\sum_{\mu=k}^{2k-1} \frac{\mu!^2}{(2k)!(2k-\mu)!^2(2\mu-2k)!}\partial_{\nu_z}N_{\mu+1}(z,\zeta), \tag{7.12}$$

*where*

$$\partial_{\nu_z}N_{\mu+1}(z,\zeta) = z\partial_z N_{\mu+1}(z,\zeta) + \overline{z}\partial_{\overline{z}}N_{\mu+1}(z,\zeta) \quad on \quad L.$$

**Proof:** Applying (7.6) to $\frac{1}{(2k-1)!^2}(|z|^2-1)^{2k-1}$, and then from (7.4), (7.10) and the normalization for $N_i$ $(1 \le i \le m)$,

$$\frac{1}{(2k-1)!^2}(|\zeta|^2-1)^{2k-1} + \frac{1}{\pi}\int\limits_{\Omega} N_{2k-1}(z,\zeta)\mathrm{d}x\mathrm{d}y = -\frac{1}{4\pi i}\sum_{\mu=0}^{2(k-1)}\int\limits_{L} \partial_{\nu_z}N_{\mu+1}(z,\zeta)(\partial_z\partial_{\overline{z}})^{\mu}\left[\frac{1}{(2k-1)!^2}(|z|^2-1)^{2k-1}\right]\frac{\mathrm{d}z}{z}. \tag{7.13}$$

Thus, expression (7.11) follows from Lemma 7.2.1 and the property that $\partial_{\nu_z}N_{\mu+1}(z,\zeta)$ is only depending on $\zeta$ for $z \in L$. Similarly, (7.12) is also true.

Therefore, from (7.4) and Lemma 7.2.3, we obtain the recursive boundary behavior for $N_m(z,\zeta)$.

**Theorem 7.2.2.** *For* $m \ge 2$ *and* $\zeta \in \Omega$,

$$\partial_{\nu_z}N_m(z,\zeta) = \begin{cases} -\sum\limits_{\mu=[\frac{m}{2}]}^{m-2} \dfrac{\mu!^2}{(m-1)!(m-1-\mu)!^2(2\mu-m+1)!}\partial_{\nu_z}N_{\mu+1}(z,\zeta) \\ \qquad -\dfrac{4n}{(m-1)!^2}(|\zeta|^2-1)^{m-1}, \quad z \in L\backslash\{1,\omega\}, \\ 0, \qquad z \in (\omega,0)\cup(0,1), \end{cases}$$

*where $n$ is defined with respect to the angle $\pi/n$ for $\Omega$ as before and*

$$\partial_{\nu_z} N_1(z,\zeta) = \begin{cases} -4n, & z \in L\backslash\{1,\omega\}, \\ 0, & z \in (\omega,0)\cup(0,1), \end{cases} \qquad for \quad \zeta \in \Omega.$$

Finally, the polyharmonic Neumann problem is studied.

**Theorem 7.2.3.** *The polyharmonic Neumann problem*

$$(\partial_z\partial_{\overline{z}})^m w = f \quad in \ \ \Omega, \quad \partial_{\nu_z}(\partial_z\partial_{\overline{z}})^\sigma w = \gamma_\sigma \quad on \ \ \partial\Omega,$$
$$\frac{n}{\pi i}\int_L (\partial_\zeta\partial_{\overline{\zeta}})^\sigma w(\zeta)\frac{\mathrm{d}\zeta}{\zeta} = c_\sigma, \quad 0 \le \sigma \le m-1,$$

*for $f \in L_p(\Omega;\mathbb{C}),\ p > 2,\ \gamma_\sigma \in C(\partial\Omega;\mathbb{C}),\ c_\sigma \in \mathbb{C}$ is solvable if and only if*

$$\begin{aligned}\frac{1}{4\pi}\int_{\partial\Omega}\sum_{\mu=\sigma+1}^{m}\gamma_{\mu-1}(\zeta)\partial_{\nu_z}N_{\mu-\sigma}(z,\zeta)\mathrm{d}s_\zeta &= \sum_{\mu=\sigma+1}^{m-1}c_\mu b_{\mu-\sigma} \\ &+\frac{1}{\pi}\int_\Omega \partial_{\nu_z}N_{m-\sigma}(z,\zeta)f(\zeta)\mathrm{d}\xi\mathrm{d}\eta, \qquad 0\le\sigma\le m-1,\end{aligned} \tag{7.14}$$

*where $\partial_{\nu_z} = z\partial_z + \overline{z}\partial_{\overline{z}}$ for $z \in L$ and*

$$b_{k-1} = -\sum_{\mu=[\frac{k}{2}]}^{k-2}\frac{\mu!^2}{(k-1)!(k-1-\mu)!^2(2\mu-k+1)!}b_\mu, \quad 3 \le k \le m,$$
$$with \ \ b_1 = -2.$$

*Then its solution is uniquely expressed as*

$$\begin{aligned} w(z) = \sum_{\mu=0}^{m-1}\Bigg\{ &-\frac{1}{4n}c_\mu\beta_{\mu+1}(z) + \frac{1}{4\pi}\int_{\partial\Omega}\gamma_\mu(\zeta)N_{\mu+1}(z,\zeta)\mathrm{d}s_\zeta\Bigg\} \\ &-\frac{1}{\pi}\int_\Omega N_m(z,\zeta)f(\zeta)\mathrm{d}\xi\mathrm{d}\eta, \quad z\in\Omega,\end{aligned} \tag{7.15}$$

*with*

$$\beta_{\mu+1}(z) = \partial_{\nu_\zeta}N_{\mu+1}(z,\zeta), \quad 0 \le \mu \le m-1 \ \ for \ \ \zeta \in L,\ z \in \Omega.$$

*Besides, it should be noted that*

$$b_\mu = \frac{1}{4n}\partial_{\nu_z}\beta_{\mu+1}(z), \quad 1 \le \mu \le m-1,$$

*with $\partial_{\nu_z} = z\partial_z + \overline{z}\partial_{\overline{z}}$ for $z \in L$.*

**Proof:** We adopt the inductive method to prove the conclusion. From Lemma 4.2.1 and Theorem 4.3.4, the conclusion is true for $m = 1,\ 2$. We assume that the solution and the solvability conditions are valid for $m \leq k-1$ $(k \geq 2)$, then the polyharmonic Neumann problem for $m = k$ can be transformed into

$$(\partial_z\partial_{\overline{z}})^{k-1}w = w_1 \ \text{ in } \ \Omega, \ \ \partial_{\nu_z}(\partial_z\partial_{\overline{z}})^{\sigma}w = \gamma_\sigma \ \text{ on } \ \partial\Omega,$$
$$\frac{n}{\pi i}\int\limits_L (\partial_\zeta\partial_{\overline{\zeta}})^{\sigma}w(\zeta)\frac{\mathrm{d}\zeta}{\zeta} = c_\sigma, \ \ 0 \leq \sigma \leq k-2,$$

and

$$\partial_z\partial_{\overline{z}}w_1 = f \ \text{ in } \ \Omega, \quad \partial_{\nu_z}w_1 = \gamma_{k-1} \ \text{ on } \ \partial\Omega, \quad \frac{n}{\pi i}\int\limits_L w_1(\zeta)\frac{\mathrm{d}\zeta}{\zeta} = c_{k-1}.$$

Then, we know if and only if

$$\frac{1}{2\pi}\int\limits_{\partial\Omega}\gamma_{k-1}(\zeta)\mathrm{d}s_\zeta = \frac{2}{\pi}\int\limits_{\Omega} f(\zeta)\mathrm{d}\xi\mathrm{d}\eta, \tag{7.16}$$

and

$$\begin{aligned}&\frac{1}{4\pi}\int\limits_{\partial\Omega}\sum_{\mu=\sigma+1}^{k-1}\gamma_{\mu-1}(\zeta)\partial_{\nu_z}N_{\mu-\sigma}(z,\zeta)\mathrm{d}s_\zeta \\ &= \sum_{\mu=\sigma+1}^{k-2}c_\mu b_{\mu-\sigma} + \frac{1}{\pi}\int\limits_{\Omega}\partial_{\nu_z}N_{k-1-\sigma}(z,\zeta)w_1(\zeta)\mathrm{d}\xi\mathrm{d}\eta, \quad 0 \leq \sigma \leq k-2,\end{aligned} \tag{7.17}$$

are satisfied, $w_1,\ w$ can be uniquely expressed as, respectively,

$$w_1(z) = c_{k-1} + \frac{1}{4\pi}\int\limits_{\partial\Omega}\gamma_{k-1}(\zeta)N_1(z,\zeta)\mathrm{d}s_\zeta - \frac{1}{\pi}\int\limits_{\Omega} f(\zeta)N_1(z,\zeta)\mathrm{d}\xi\mathrm{d}\eta, \tag{7.18}$$

and

$$\begin{aligned}w(z) = \sum_{\mu=0}^{k-2}\Bigg\{ -\frac{1}{4n}c_\mu\beta_{\mu+1}(z) + \frac{1}{4\pi}\int\limits_{\partial\Omega}\gamma_\mu(\zeta)N_{\mu+1}(z,\zeta)\mathrm{d}s_\zeta\Bigg\} \\ -\frac{1}{\pi}\int\limits_{\Omega}N_{k-1}(z,\zeta)w_1(\zeta)\mathrm{d}\xi\mathrm{d}\eta.\end{aligned} \tag{7.19}$$

Putting (7.18) in the area integral of (7.17) gives

$$\frac{1}{\pi}\int_{\Omega}\partial_{\nu_z}N_{k-1-\sigma}(z,\zeta)w_1(\zeta)\mathrm{d}\xi\mathrm{d}\eta = \frac{c_{k-1}}{\pi}\int_{\Omega}\partial_{\nu_z}N_{k-1-\sigma}(z,\zeta)\mathrm{d}\xi\mathrm{d}\eta$$
$$-\frac{1}{4\pi}\int_{\partial\Omega}\gamma_{k-1}(\zeta)\partial_{\nu_z}N_{k-\sigma}(z,\zeta)\mathrm{d}s_\zeta + \frac{1}{\pi}\int_{\Omega}f(\zeta)\partial_{\nu_z}N_{k-\sigma}(z,\zeta)\mathrm{d}\xi\mathrm{d}\eta,$$

thus from

$$\partial_{\nu_z}\partial_{\nu_\zeta}N_{k-\sigma}(z,\zeta) = \frac{4n}{\pi}\int_{\Omega}\partial_{\nu_z}N_{k-1-\sigma}(z,\zeta)\mathrm{d}\xi\mathrm{d}\eta, \quad 0\le\sigma\le k-2,$$
$$\text{with } \partial_{\nu_z} = z\partial_z + \overline{z}\partial_{\overline{z}}, \quad z,\ \zeta\in L,$$

we have

$$\frac{1}{4\pi}\int_{\partial\Omega}\sum_{\mu=\sigma+1}^{k}\gamma_{\mu-1}(\zeta)\partial_{\nu_z}N_{\mu-\sigma}(z,\zeta)\mathrm{d}s_\zeta = \sum_{\mu=\sigma+1}^{k-1}c_\mu b_{\mu-\sigma}$$
$$+\frac{1}{\pi}\int_{\Omega}\partial_{\nu_z}N_{k-\sigma}(z,\zeta)f(\zeta)\mathrm{d}\xi\mathrm{d}\eta, \quad 0\le\sigma\le k-2. \tag{7.20}$$

Then, (7.20) and (7.16) imply that (7.14) is true for $m = k$. Similarly, (7.15) is also valid. This completes the proof.

# Appendix A: The Tetra-harmonic Green Function for the Unit Disc

Similarly to the tetra-harmonic Green function $\widehat{G}_4(z,\zeta)$ constructed in (6.19) for the fan-shaped domain $\Omega$ with angle $\pi/n$ ($n \in \mathbb{N}$), we also establish the tetra-harmonic Green function $\widetilde{G}_4(z,\zeta)$ explicitly for the unit disc by convolution, that is,

$$
\begin{aligned}
\widetilde{G}_4(z,\zeta) &= \frac{1}{36}|\zeta - z|^6 \log\left|\frac{1-z\overline{\zeta}}{\zeta - z}\right|^2 - \frac{1}{36}(|z|^2-1)(|\zeta|^2-1)(\overline{z}^2\zeta^2 + z^2\overline{\zeta}^2 - 30)\\
&+\frac{1}{12}(|z|^2-1)(|\zeta|^2-1)(|\zeta|^2+|z|^2)(\overline{z}\zeta + z\overline{\zeta} - 4) - \frac{1}{72}(|z|^4-1)(|\zeta|^4-1)(\overline{z}\zeta + z\overline{\zeta})\\
&+\frac{1}{36}(|z|^6-1)(|\zeta|^6-1)\left\{\frac{1}{2!}\left(\frac{1}{\overline{z}\zeta} + \frac{1}{z\overline{\zeta}}\right) + \frac{1}{(\overline{z}\zeta)^2} + \frac{1}{(z\overline{\zeta})^2}\right.\\
&\qquad\left. + \frac{\log(1-\overline{z}\zeta)}{(\overline{z}\zeta)^3} + \frac{\log(1-z\overline{\zeta})}{(z\overline{\zeta})^3}\right\}\\
&-\frac{1}{12}\Big[(|z|^6-1)(|\zeta|^4-1) + (|z|^4-1)(|\zeta|^6-1) + 3(|z|^2-1)(|\zeta|^4-1)\\
&\qquad +3(|z|^4-1)(|\zeta|^2-1)\Big] \times \left\{\frac{\log(1-\overline{z}\zeta)}{(\overline{z}\zeta)^2} + \frac{\log(1-z\overline{\zeta})}{(z\overline{\zeta})^2} + \frac{1}{\overline{z}\zeta} + \frac{1}{z\overline{\zeta}}\right\}\\
&+\frac{1}{12}\Big[(|z|^6-1)(|\zeta|^2-1) + (|z|^2-1)(|\zeta|^6-1) + 3(|z|^4-1)(|\zeta|^4-1)\\
&\qquad +3(|z|^2-1)(|\zeta|^2-1)\Big] \times \left[\frac{\log(1-\overline{z}\zeta)}{\overline{z}\zeta} + \frac{\log(1-z\overline{\zeta})}{z\overline{\zeta}}\right]\\
&\quad -(|z|^2-1)(|\zeta|^2-1)\sum_{l=0}^{\infty}\frac{1}{(l+1)^3}\left[(z\overline{\zeta})^l + (\overline{z}\zeta)^l\right]\\
&\quad +\frac{1}{2}(|z|^2-1)(|\zeta|^2-1)(|z|^2+|\zeta|^2+1)\sum_{l=0}^{\infty}\frac{1}{(l+1)^2}\left[(z\overline{\zeta})^l + (\overline{z}\zeta)^l\right], \quad z,\zeta \in \mathbb{D},
\end{aligned}
$$

with the unit disc $\mathbb{D} = \{z,\ |z| < 1\}$. We can exactly confirm that $\widetilde{G_4}(z, \zeta)$ above satisfies

$$\begin{cases} \partial_z \partial_{\overline{z}} \widetilde{G_4}(z, \zeta) = \widetilde{G_3}(z, \zeta), & z,\ \zeta \in \mathbb{D}, \\ \widetilde{G_4}(z, \zeta) = 0, & z \in \partial\mathbb{D},\ \zeta \in \mathbb{D}, \end{cases}$$

where the tri-harmonic Green function $\widetilde{G_3}(z, \zeta)$ for the domain $\mathbb{D}$ is, (see [29])

$$\begin{aligned} \widetilde{G_3}(z, \zeta) \ &= \frac{1}{4}|\zeta - z|^4 \log \left| \frac{1 - z\overline{\zeta}}{\zeta - z} \right|^2 + \frac{1}{4}(1 - |z|^2)(1 - |\zeta|^2)(z\overline{\zeta} + \overline{z}\zeta - 4) \\ &- \frac{1}{4}(1 - |z|^4)(1 - |\zeta|^4) \left[ \frac{\log(1 - z\overline{\zeta})}{(z\overline{\zeta})^2} + \frac{\log(1 - \overline{z}\zeta)}{(\overline{z}\zeta)^2} + \frac{1}{z\overline{\zeta}} + \frac{1}{\overline{z}\zeta} \right] \\ &+ \frac{1}{2}(1 - |z|^2)(1 - |\zeta|^2)(|z|^2 + |\zeta|^2) \left[ \frac{\log(1 - z\overline{\zeta})}{z\overline{\zeta}} + \frac{\log(1 - \overline{z}\zeta)}{\overline{z}\zeta} \right] \\ &+ (1 - |z|^2)(1 - |\zeta|^2) \sum_{l=0}^{\infty} \frac{1}{(l + 1)^2} \left[ (z\overline{\zeta})^l + (\overline{z}\zeta)^l \right]. \end{aligned}$$

Hence, obviously $\widetilde{G_4}(z, \zeta)$ is just the tetra-harmonic Green function for the unit disc $\mathbb{D}$ by convolution.

# Appendix B: The Tri-harmonic Neumann Function for the Unit Disc

In [29], a tri-harmonic Neumann function for the unit disc is constructed by convolution. However, the expression of the tri-harmonic Neumann function $\widetilde{N_3}(z,\zeta)$ is given via integral representation. As a matter of fact, it can be established via elementary functions explicitly. That is,

$$\begin{aligned}
\widetilde{N_3}(z,\zeta) &= \frac{3}{2}(|\zeta|^4+|z|^4)+5(|\zeta|^2+1)(|z|^2+1)+2(|\zeta|^2+|z|^2+6)\\
&\quad+\frac{1}{4}(|z|^2-1)(|\zeta|^2-1)(\overline{z}\zeta+z\overline{\zeta})+\frac{1}{4}|\zeta-z|^4\log\left|\frac{1-z\overline{\zeta}}{\zeta-z}\right|^2\\
&\quad-\left[2(|z|^2+2)(|\zeta|^2+2)+\frac{1}{2}(|\zeta|^4+|z|^4)\right]\log|1-z\overline{\zeta}|^2\\
&\quad+\left[\frac{1}{2}(|\zeta|^2+|z|^2)(|z|^2+1)(|\zeta|^2+1)+4(|z|^2+|\zeta|^2+2)\right]\\
&\qquad\times\left[\frac{\log(1-\overline{z}\zeta)}{\overline{z}\zeta}+\frac{\log(1-z\overline{\zeta})}{z\overline{\zeta}}\right]\\
&\quad-\frac{1}{4}(|z|^4+1)(|\zeta|^4+1)\left[\frac{\log(1-\overline{z}\zeta)}{(\overline{z}\zeta)^2}+\frac{\log(1-z\overline{\zeta})}{(z\overline{\zeta})^2}+\frac{1}{\overline{z}\zeta}+\frac{1}{z\overline{\zeta}}\right]\\
&\quad+\sum_{l=0}^{\infty}\left\{\left[\frac{8}{(l+1)^3}-\frac{(|\zeta|^2+1)(|z|^2+1)}{(l+2)^2}-\frac{4|z|^2+4|\zeta|^2+6}{(l+1)^2}\right]\right.\\
&\qquad\left.\times\left[(\overline{z}\zeta)^{l+1}+(z\overline{\zeta})^{l+1}\right]\right\},\quad z,\zeta\in\mathbb{D}.
\end{aligned}$$

By computation, we can verify that $\widetilde{N_3}(z,\zeta)$ has the properties,

$$\partial_z\partial_{\overline{z}}\widetilde{N_3}(z,\zeta)=\widetilde{N_2}(z,\zeta),\quad z\in\mathbb{D},$$

$$\partial_{\nu_z}\widetilde{N_3}(z,\zeta)=-\left[\frac{1}{2}(1-|\zeta|^2)^2+(1-|\zeta|^2)\right],\quad z\in\partial\mathbb{D},\ \zeta\in\mathbb{D}.$$

Further, the normalization condition holds,

$$\frac{1}{2\pi i}\int_{\partial\mathbb{D}}\widetilde{N_3}(z,\zeta)\frac{\mathrm{dz}}{z}=0,$$

where the harmonic Neumann function $\widetilde{N_1}(z,\zeta)$ and the biharmonic Neumann function $\widetilde{N_2}(z,\zeta)$ for the unit disc are determined in [11, 29],

$$\widetilde{N_1}(z,\zeta) = -\log|(\zeta - z)(1 - z\overline{\zeta})|^2, \quad z,\ \zeta \in \mathbb{D},$$

and

$$\begin{aligned}\widetilde{N_2}(z,\zeta) = |\zeta - z|^2 \left[4 - \log|(\zeta - z)(1 - z\overline{\zeta})|^2\right] - 2(z\overline{\zeta} + \overline{z}\zeta)\log|1 - z\overline{\zeta}|^2 \\ - \sum_{l=1}^{+\infty} \frac{4}{(l+1)^2}\left[(z\overline{\zeta})^{l+1} + (\overline{z}\zeta)^{l+1}\right] \\ +(|\zeta|^2 + 1)(|z|^2 + 1)\left[\frac{\log(1 - \overline{z}\zeta)}{\overline{z}\zeta} + \frac{\log(1 - z\overline{\zeta})}{z\overline{\zeta}}\right], \quad z,\ \zeta \in \mathbb{D}.\end{aligned}$$

Therefore, by the properties of $\widetilde{N_3}(z,\zeta)$, we easily know that it is just the desired tri-harmonic Neumann function for the unit disc $\mathbb{D}$ by convolution.

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# Zusammenfassung

In dieser Dissertation werden einige Randwertprobleme für komplexe partielle Differentialgleichung in Kreissektoren untersucht. Zuerst wird die Schwarz-Poisson Integraldarstellung in Kreissektoren mit dem Öffnungswinkel $\pi/n$ ($n \in \mathbb{N}$) mit Hilfe der Reflektionsmethode bewiesen und die entsprechenden Schwarz und Dirichlet Probleme studiert. Danach wird die Schwarz-Poissonsche Integraldarstellung mit Hilfe einer geeigneten konformen Abbildung auf allgemeine Kreissektoren mit Öffnungswinkel $\pi/\alpha$ ($\alpha \geq 1/2$) erweitert. Damit werden die Schwarz und Dirichlet Probleme für die Cauchy-Riemann Gleichung gelöst. Es wird eine Brücke zu den entsprechenden Formeln im Einheitskreis und dem Kreissektor mit $\alpha = 1/2$ geschlagen, und die Schwarz-Poisson Formel für den Einheitskreis aus der Schwarz-Poisson Formel für $\alpha = 1/2$ hergeleitet.

Eine harmonische Greensche Funktion und eine harmonische Neumannsche Funktion werden für Kreissektoren mit Winkel $\pi/\alpha$ ($\alpha \geq 1/2$), konstruiert und damit die Dirichlet und Neumann Probleme für die Poisson Gleichung behandelt. Insbesondere wird die äußere Richtungsableitung in den drei Eckpunkten des Gebietes in geeigneter Weise definiert. Eine biharmonische Greensche Funktion, eine biharmonische Neumannsche Funktion, eine triharmonische Greensche Funktion, eine triharmonische Neumannsche Funktion und eine terta-harmonische Greensche Funktion werden für die Kreissektoren mit Öffnungswinkel $\pi/n$ ($n \in \mathbb{N}$) in expliziter Form konstruiert. Darüber hinaus wird aufgezeigt, wie sich eine tetraharmonische Neumann Funktion gewinnen lässt, und ein Ausdruck für diese Funktion wird mit Hilfe einer Integraldarstellung angegeben. Die zugehörigen Dirichlet und Neumann Probleme werden gelöst.

Schließlich wird ein Iterationsprozess zur Behandlung von polyharmonischen Dirichlet und Neumann Problemen für die Poissonsche Gleichung höherer Ordnung in Kreissektoren mit Winkel $\pi/n$ ($n \in \mathbb{N}$) und zugehörige Lösbarkeitsbedingungen angegeben. Auch wird das Randverhalten der gefalteten polyharmonischen Greenschen und Neumannschen Funktionen ausführlich untersucht. In einem Anhang

werden eine tetraharmonische Greensche Funktion und eine triharmonische Neumann Funktion für den Einheitskreis explizit konstruiert.

**Stichwörter**: Schwarz-Poisson Darstellung, polyharmonische Green Funktion, polyharmonische Neumann Funktion, Schwarz Problem, Dirichlet Problem, Neumann Problem.

# Curriculum Vitae