Triangles in Euclidean Arrangements

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Abstract. The number of triangles in arrangements of lines and pseudolines has been object of some research. Most results, however, concern arrangements in the projective plane. In this article we add results for the number of triangles in Euclidean arrangements of pseudolines. Though the change in the embedding space from projective to Euclidean may seem small there are interesting changes both in the results and in the techniques required for the proofs.

In 1926 Levi proved that a nontrivial arrangement -simple or not- of n pseudolines in the projective plane contains n triangles. To show the corresponding result for the Euclidean plane, namely, that a simple arrangement of n pseudolines contains $n-2$ triangles, we had to find a completely different proof. On the other hand a non-simple arrangements of n pseudolines in the Euclidean plane can have as few as $2n/3$ triangles and this bound is best possible. We also discuss the maximal possible number of triangles and some extensions.

Mathematics Subject Classifications (1991). 52A10, 52C10.

Key Words. Arrangement, Euclidean plane, pseudoline, strechability, triangle.

1 Introduction, Definitions and Overview

The number p_3 of triangles in arrangements of (pseudo)lines has been object of some research. In this article we add new results concerning the number of triangles in Euclidean arrangements of pseudolines.

Grünbaum [Grü72] defines an *arrangement $\mathcal{A}$ of lines* as a finite collection $\{L_0, L_1, \ldots, L_n\}$ of lines, i.e., 1–dimensional subspaces in the real projective plane $\mathbb{P}$. Specifying a line L_0 in $\mathcal{A}$ as the "line at infinity" induces the arrangement $\mathcal{A}_{L_0}$ of lines $\{L_1, \ldots, L_n\}$ in the Euclidean plane $\mathbb{E} = \mathbb{P} \setminus L_0$.

With an arrangement we associate the cell complex of vertices edges and cells into which the lines of the arrangement decompose the underlying space $\mathbb{P}$ or $\mathbb{E}$. Arrangements are *isomorphic* provided their cell complexes are isomorphic.

An *arrangement $\mathcal{B}$ of pseudolines* in $\mathbb{P}$ is a collection $\{P_0, P_1, \ldots, P_n\}$ of simple closed curves (we call them *pseudolines*) in $\mathbb{P}$ such that every two curves have exactly one point in common. Specifying a pseudoline P_0 in $\mathcal{B}$ as the line at infinity induces the arrangement $\mathcal{B}_{P_0}$ of pseudolines $\{P_1, \ldots, P_n\}$ in $\mathbb{P} \setminus P_0$. Since $\mathbb{P} \setminus P_0$ is homeomorphic to the Euclidean plane and we are interested in properties of the induced cell complex we may regard $\mathcal{B}_{P_0}$ as an arrangement in $\mathbb{E}$.

Already in early work of Levi [Lev26] and Ringel [Rin56] it has been noted that arrangements of pseudolines are a proper generalization of arrangements

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of lines. This is due to the existence of incidence laws in plane geometry ,e.g., the Theorem of Pappus. Arrangements of pseudolines have gained attention since they provide a generic model for oriented matroids of rank 3. In this context questions of strechability have attained considerable interest. For more about these connections we refer the reader to the 'bible of oriented matroids' [BLS+93].

An arrangement is *trivial* if there exists a point common to all (pseudo)lines. If no point belongs to more then two of the (pseudo)lines we call the arrangement *simple*.

Euclidean arrangements of pseudolines will be the main object of investigations in this paper. Work with these objects is simplified by the fact that every arrangement of pseudolines, i.e., of doubly unbounded curves, is isomorphic to an arrangement of x-monotone pseudolines, i.e., of curves that intersect every vertical line in exactly one point. Particularly nice pictures of Euclidean arrangements of pseudolines are given by their *wiring diagrams* introduced in Goodman [Goo80], see Figure 1. In this representation the n x-monotone curves are restricted to n y-coordinates except for some local switches where adjacent lines cross. Knuth [Knu92] points out a connection with 'primitive sorting networks'.

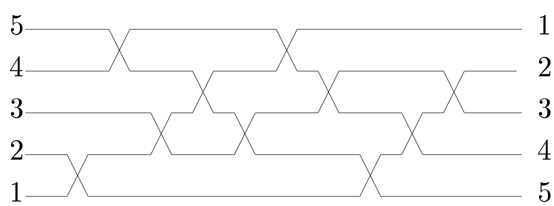

Figure 1. Wiring diagram of a simple arrangement of 5 pseudolines.

We now summarize bounds for the number p_3 of triangles in arrangements.

Theorem 1 *For every arrangement $\mathcal{A}$ of n pseudolines in $\mathbb{P}$:*

(1) *Every pseudoline is incident with at least three triangles. Since every triangle is incident with three lines this implies $p_3 \geq n$.*

(2) $p_3 \leq \frac{1}{3}n(n-1)$ *for $n \geq 10$ with equality for infinitely many values of n.*

Part (1) is due to Levi [Lev26]. The lower bound for p_3 it best possible. To see this take the n supporting lines of the edges of a regular n-gon for $n \geq 4$. The arrangement thus obtained is a simple arrangement of lines with $p_3 = n$.

Part (2) has a more entangled history. In [Grü72] the following easy argument for $p_3 \leq \frac{1}{3}n(n-1)$ in simple arrangements is found. If $\mathcal{A}$ is simple then only one of the cells bounded by an edge can be a triangle. Since there are $n(n-1)$ edges and every triangle uses three of them the bound is established. Grünbaums conjectured the same bound for nonsimple arrangements of lines with sufficiently large n. Several special cases and lower bounds where proved by Strommer, Purdy and others. Finally Roudneff [Rou96] proved the conjectured bound for $n \geq 10$. By perturbing high degree vertices so that suitable arrangements are formed in the neighborhood he shows that p_3 is maximized by what he calls 'reduced arrangements'. In particular these arrangements have no vertices of degree more then four. The crucial part of the proof is to show that if t_i counts vertices of degree i then for $n \geq 9$ every reduced arrangement has

$$3p_3 \leq 2(t_2 + 3t_3 + 6t_4).$$

Since $\sum_k \binom{k}{2} t_k = \binom{n}{2}$ this implies the bound.

Infinite families of simple arrangements with $p_3 = \frac{1}{3}n(n-1)$ have been obtained by Roudneff [Rou86] and Harborth [Har85]. For stretchable arrangements the best known constructions are due to Füredi and Palesti [FP84]. Their examples have at least $\frac{1}{3}n(n-3)$ triangles.

In this paper we discuss triangles in Euclidean arrangements. The cell complex of an arrangement in $\mathbb{E}$ consists of unbounded and bounded cells. In our treatment we ignore unbounded cells. In the arrangement of Figure 1 we thus count 3 triangles and 3 quadrangles. Our main results are summarized in the following Theorem whose proof will be given in sections 2 and 3.

Theorem 2 *For every arrangement $\mathcal{B}$ of n pseudolines in $\mathbb{E}$:*

(1) *If $\mathcal{B}$ is simple then $p_3 \geq n - 2$.*

(2) *If $n \geq 6$ then $p_3 \geq \frac{2}{3}n$ with equality for all $n = 0 \pmod 3$.*

(3) *$p_3 \leq \frac{1}{3}n(n-2)$ with equality for infinitely many values of n.*

Part (1) again has a long history. Roberts 1889 claimed that for every simple arrangement $\mathcal{A}$ of $n+1$ lines in $\mathbb{P}$ and every line L of $\mathcal{A}$ there are $n-2$ triangles not incident with L. The argument however was considered non-convincing. Ninety years later Shannon [Sha79] proved Roberts theorem, actually, he proved the analog of Roberts theorem for arbitrary dimensions. In particular this implies that every stretchable arrangement $\mathcal{B}$ of n lines in $\mathbb{E}$ has at least $n-2$ triangles. Add the line at infinity to obtain a projective arrangement and apply Roberts theorem.

Shannon's proof does not require that the arrangement is simple. Therefore, Shannon's theorem together with Theorem 2 (2) gives the following amazing result.

Corollary 3 *The count of triangles can be a certificate for nonstrechability of nonsimple Euclidean arrangements.*

A similar effect in the projective setting was conjectured by Grünbaum and proved by Roudneff [Rou88]. A nonsimple projective arrangement with $p_3 = n$ is nonstrechable. An Example of such an arrangement is due to Canham, see Grünbaum [Grü72, page 55]. In Section 3 we describe a family W_n of arrangements with few triangles. If W_n is considered as an arrangement in the projective plane it is a nonsimple arrangement with n lines and $p_3 = n$.

It is interesting to note that Levi's theorem about the number of triangles incident to a line and Roberts respectively Shannons theorem about the number of triangles avoiding a line both give easy double-counting proofs for $p_3 \geq n$. We elaborate the second:

Corollary 4 *The number of triangles in a simple arrangement $\mathcal{A}$ of n pseudolines in $\mathbb{P}$ is at least n.*

Proof. For each pseudoline P_i consider the Euclidean arrangement $\mathcal{A}_{P_i}$ obtained by taking P_i as line at infinity. Each such arrangement has at least $(n-1)-2$ triangles. Altogether this gives at least $n(n-3)$ triangles. Any fixed triangle Δ in $\mathcal{A}$ is bounded by three pseudolines and hence counted exactly $n-3$ times. This shows that there are at least n different triangles. □

The upper bound on the number of triangles in the Euclidean case claimed in (3) of Theorem 2 can be proved along the lines of Roudneff's upper bound for the projective case. The proof is long and the changes necessary for to adopt it to the Euclidean case obvious. Therefore, we will refrain from elaborating on it and refer to Roudneff's original paper [Rou96].

To show that the bound is best possible again the examples from the same paper [Rou96] do the work. Roudneff shows that there is an infinite family of simple projective arrangements with $n+1$ lines and $(n+1)n/3$ triangles. Each line of such an arrangement is incident to n triangles. Choose an arbitrary line l as line in infinity. The remaining Euclidean arrangement of n lines has $(n+1)n/3 - n = n(n-2)/3$ triangles.

2 Simple Euclidean arrangements

In this section we prove the lower bound for the number of triangles in simple arrangements in $\mathbb{E}$.

Proposition 2.1 *$p_3 \geq n-2$ for every simple arrangement $\mathcal{B}$ of n pseudolines in $\mathbb{E}$.*

Proof. We consider the finite part of $\mathcal{B}$ as a planar graph. Let V be the number of vertices E be the number of edges and F be the number of (finite!) faces. These statistics can all be expressed as functions of the number of pseudolines.

$$V = \binom{n}{2}, \qquad E = n\,(n-2), \qquad F = \binom{n-1}{2}$$

Note that in this setting Euler's formula gives $V - E + F = 1$.

We assign labels $\oplus$ or $\ominus$ to each side of every edge. Let f be one of the two (possibly unbounded) faces bounded by e and let e' and e'' be the edge-neighbors of e along f. Let l, l' and l'' be the supporting pseudolines of e, e' and e'' respectively. The label of e on the side of f is $\oplus$ if f is contained in the finite triangle T of the arrangement $\{l, l', l''\}$ otherwise the label is $\ominus$. See Figure 2 for an illustration of the definition and Figure 3 for a complete labeling. With the next lemmas we collect important properties of the edge labeling.

Lemma 5 *Every edge e of a simple arrangement has a $\oplus$ and a $\ominus$ label.*

Proof. Let f_1 and f_2 be the two faces bounded by e and let e_1', e_1'' and e_2', e_2'' be the edge-neighbors of e in these two faces. Since the arrangement is simple the supporting lines $\{l'_1, l''_1\}$ of both pairs of edges are the same. The finite triangular region T of the arrangement $\{l, l', l''\}$ has edge e on its boundary. Therefore, exactly one of the two faces f_1 and f_2 is contained in T. $\triangle$

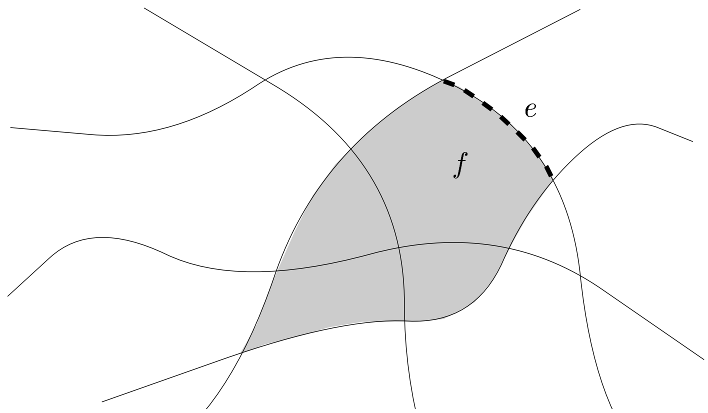

Figure 2. The label of e at f is $\oplus$ since f is contained in the shaded triangle.

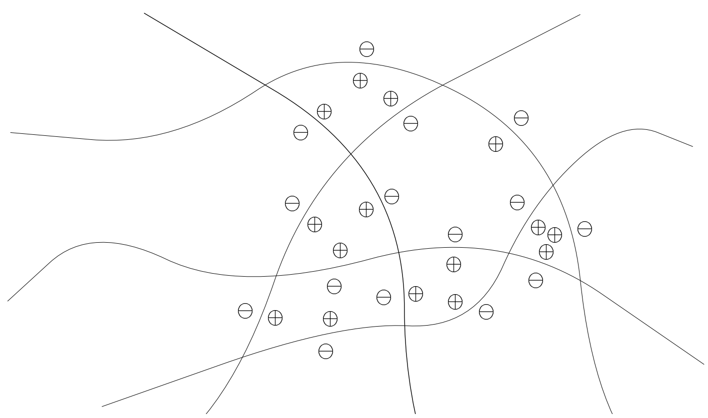

Figure 3. The arrangement of Figure 2 with the completed edge labeling.

As seen in the proof of the lemma the triangular region T used to define the edge label of e on the side of f is independent of f. This allows to adopt the notation $T(e)$ for this region.

Lemma 6 *All three edge labels in a triangle are* $\oplus$ *. A quadrangle contains two* $\oplus$ *and two* $\ominus$ *labels. For* $k \geq 5$ *a* k *sided face contains at most two* $\oplus$ *labels.*

Proof. If f is a triangle then for each of its edges e the triangular region $T(e)$ is f itself.

Let f be a quadrangle and e, $\overline{e}$ be a pair of opposite edges of f. Both edges have the same neighboring edges, hence, two of the lines bounding the triangles $T(e)$ and $T(\overline{e})$ are equal. It is easy to see that either $T(e) = f \cup T(\overline{e})$ or $T(\overline{e}) = f \cup T(e)$. In the first case e has label $\oplus$ and $\overline{e}$ has label $\ominus$ in the second case the labels are exchanged. The second pair of opposite edges also has one label $\oplus$ and the other $\ominus$.

Let f be a face with $k \geq 5$ sides the lemma immediately follows from the following

Claim. *Any two edges with label $\oplus$ in f are neighbors, i.e., share a common vertex.*

Let $e_1, e_2, \ldots, e_k$ be the edges of f numbered in counterclockwise direction along f and let l_i be the supporting line of e_i. Let e_1 have label $\oplus$ and consider an edge e_i with $3 \leq i \leq k-1$. We will show that the label of e_i is $\ominus$. The argument as given applies to the case $4 \leq i \leq k-1$, the remaining case situation $i = 3$, however is symmetric to $i = k-1$.

Face f is contained in $T(e_1)$ and line l_i has to leave $T(e_1) \setminus f$ through l_k and l_2. Figure 4 is a generic sketch of the situation.

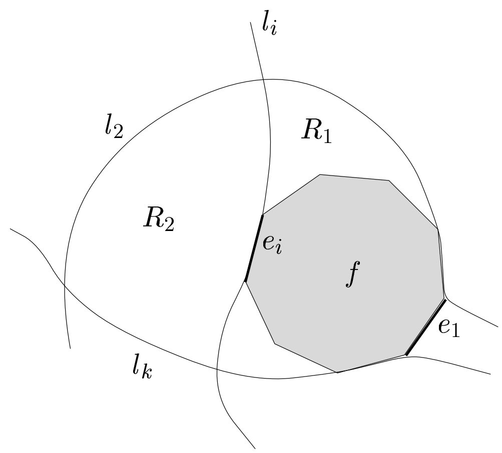

Figure 4. Edge e_1 has label $\oplus$ in f so e_i must have $\ominus$.

Consider line l_{i-1}. This line enters the region R_1 bounded by l_2, l_i and the chain of edges $e_3, e_4, \ldots, e_{i-1}$ at the vertex $e_{i-1} \cap e_i$. To leave region R_1 line l_{i-1} has to cross l_2. Therefore, l_{i-1} has to leave the region R_2 bounded by l_i, l_2 and l_k through l_k. Symmetrically, l_{i+1} has a crossing with l_k to leave the region bounded by l_k, l_i and the chain of edges $e_{i+1}, e_{i+2}, \ldots, e_k$. Therefore, to leave region R_2 line l_{i+1} has to cross l_2. This shows that l_{i-1} and l_{i+1} cross inside region R_2. Hence, $T(e_i)$ is contained in R_2 and e_i has label $\ominus$ in f.

Since e_1 was an arbitrary $\oplus$ labeled edge in F we have shown the claim. $\triangle$

We use the two lemmas to count the number of $\oplus$ labels in different ways:

$$E = \sum_f \#\{+ \text{ labels in } f\} \leq 2F + p_3.$$

With $E = n(n-2)$ and $2F = (n-1)(n-2)$ this implies

$$p_3 \geq n - 2.$$

$\square$

3 Nonsimple Euclidean arrangements

We now come to the lower bound for the number of triangles in the nonsimple case.

Proposition 2.2 *A Euclidean nonsimple and nontrivial arrangement of $n \geq 6$ pseudolines has at least $2n/3$ triangles with equality for all $n = 0 \pmod 3$.*

Proof. We distinguish two cases. First suppose that every line l of the arrangement contains crossings of the arrangement in both open halfspaces it defines. Consider l as a state of a sweepline going across the arrangement. From the theory of sweeps for arrangements of pseudolines (see e.g. [SH89]) we know that the sweep can make progress both in the forward as well as in the backward direction. A progress-move pulls line l across a crossing c of some lines of the arrangement with the property that the portion of all lines contributing to c between c and l are free of further crossings, i.e. are edges of the cell complex induced by the arrangement. Hence such a move pulls l across some triangles with corner c and an edge on l. This shows that l contributes to at least one triangle on either side. Since we assumed that every line has crossings on either side this accounts for $2n$ triangles each counted at most three times and the claim is proved in this case.

Now assume that there is a line l so that all crossings of the arrangement not on l are on one side of l. If taking away l all lines cross in just one point c then there are $n-2$ triangles in the arrangement and since we assume $n \geq 6$ we are done. Else removing l from the arrangement we still have a nontrivial arrangement which by induction has at least $2(n-1)/3$ triangles. Since l can make a sweep move to one of its sides there is at least one triangle with an edge on l that disappeared after removal of l (it turned into an unbounded region). His makes a total of $2(n-1)/3 + 1 > 2n/3$ triangles in the initial arrangement.

It remains to describe a family W_n of arrangements with $3n$ lines but only $2n$ triangles. A drawing of W_4 is given in Figure 5.

Let P be a regular $2n$-gon with edges $e_1, e_2, \dots, e_{2n}$ in counterclockwise ordering and barycenter c. Let lines $l_1, \dots, l_{2n}$ be straight lines such that l_i contains edge e_i of P. Orient the lines such that P is to their left. Note that l_i is crossed by lines $l_{i+n+1}, l_{i+n+2}, \dots, l_{i-1}, l_{i+1}, l_{i+2}, \dots l_{i+n-1}$ in this order with indices being taken cyclically. The arrangement $\mathcal{A}$ formed by these $2n$ lines has $2n$ triangles all adjacent to P. All the other faces of the arrangement are quadrangles.

For every pair l_i, l_{i+n} of parallel lines we construct an additional line g_i. We lead g_1 from the unbounded region between the positive end of l_1 and the negative end of l_n to the unbounded region between the positive end of l_{n+1} and the negative end of l_{2n}. The first line crossed by g_1 is l_1. Parallel to l_{n+1} line g_1 crosses $l_2, l_3, \dots, l_{n-1}$ and splits quadrangles into two. Before entering P line g_1 splits the triangle sitting over edge e_n into a quadrangle and a triangle. From edge e_n line g_1 joins to point c and then to the opposite edge e_{2n} to cross lines $L_{2n}, l_{2n-1}, \dots, l_{n+1}$ in this order.

Define lines $g_2, \dots, g_n$ by rotational symmetry and note that all g_i cross in c. The arrangement $\mathcal{A} \cup \{g_1, g_2, \dots, g_n\}$ has the same number of triangles as $\mathcal{A}$.

So far we still have n pairs of parallel lines. Note however that without increasing the number of triangles we may arbitrarily choose to have the crossing of pair $\{l_i, l_{i+n}\}$ to be on the side of the positive end of either l_i or l_{i+n}. Thus W_n is itself not just one but an exponentially large class of examples. □

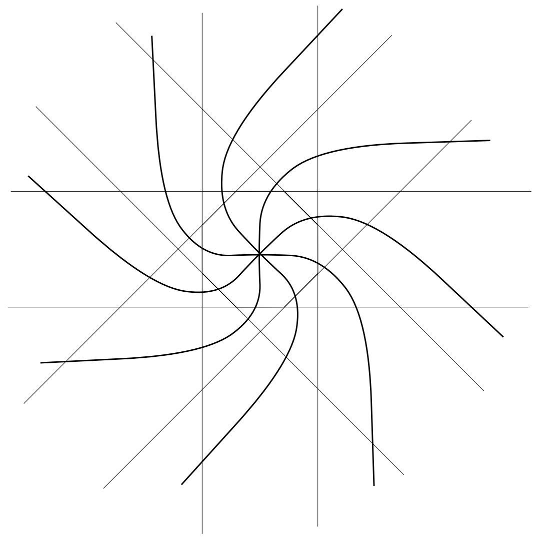

Figure 5. The arrangement W_4 with 12 lines and 8 triangles.

4 Triangles in arrangements with multiple intersections

In his monograph Grünbaum extends the notion of arrangements in several directions. Let an *arrangement of pseudocircles* be a family of closed curves with the property that any two curves cross twice*. A *digon* in such an arrangement is a face bounded by only two of the curves. Grünbaum asks for the relationship between the number of triangles and digons in such arrangements. In particular he conjectures [Grü72, Conjecture 3.7] that every digon-free arrangement of pseudocircles contains $2n - 4$ triangles. The only progress on this conjecture is a result of Snoeyink and Hershberger [SH89]. They prove $p_3 \geq 4n/3$. The proof is only given for the simple case, i.e., no three curves cross in a single point. However, it is not hard to see that it also applies to the general case.

Based on the arrangements W_n from Section 3 it is possible to construct examples of nonsimple arrangements of pseudocircles in $\mathbb{P}$ with only $4n/3$ triangles. The idea is to glue two copies of W_n together such that all faces generated by gluing are quadrangles, see Figure 6. Hence, the result of Snoeyink and Hershberger is best possible. However, if the arrangement is simple, i.e., no three curves meet in a single point we think that Grünbaum's conjecture should proof correct. For emphasis we restate the conjecture.

*Grünbaum calls this an *arrangement of curves*

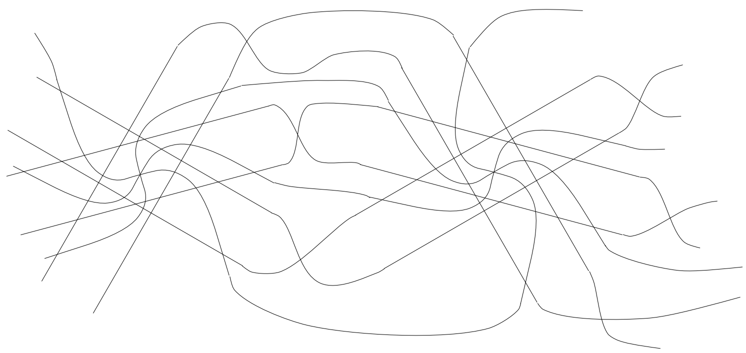

Figure 6. A digon-free arrangement of 9 two-intersecting curves and 12 triangles.

Conjecture 1 *For every simple digon-free arrangement of pseudocircles* $p_3 \geq 2n - 4$.

We feel that the spirit of Euclidean arrangements is captured well with the following generalization. Call an arrangement of x-monotone curves with the property that any two curves cross exactly k times a k-curve arrangement. Again based on the family W_n it is possible to obtain k-curve arrangements of n curves with only $2kn/3$ triangles. On the other hand we conjecture.

Conjecture 2 *Every simple digon-free* k*-curve arrangement contains at least* $k(n-2)$ *triangles.*

If true this would obviously be best possible since gluing together k appropriate arrangements of pseudolines with $n-2$ triangles each gives arrangements with only $k(n-2)$ triangles.

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