Chapter 2

Robot's spatial localization

2.1 Introduction

To locate a robot in space it is necessary to have a mathematical tool that allows the space localization of its points. In one plane, the positioning has two degrees of freedom, therefore a point's position will defined by two independent components. In the case of a three-dimensional space it will be necessary to use three components, and so on.

In this section the rotation matrix concept is introduced, also the homogeneous transformation matrix and its composition, to find the robot's localization, by several simple transformations resolved in matrix algebra.

2.2 Position

The more intuitive form used to specifying the position of a point is the cartesian coordinate system, there are also other methods like the polar coordinates for two dimensions and the cylindrical and spherical of three dimensions [20].

The reference systems are usually defined by perpendicular axes between them with a defined origin. These are denominated cartesian systems, and in the case of working in the plane (2 dimensions), the reference system corresponding OXY is defined by two coordinated vectors OX and OY perpendicular between them with a point O of common intersection, as shown in Figure 2.1 left).

For the three dimensions space, cartesian system $OXYZ$ is compound by three orthonormal vectors of coordinate OX, OY, OZ, as shown in Figure 2.1
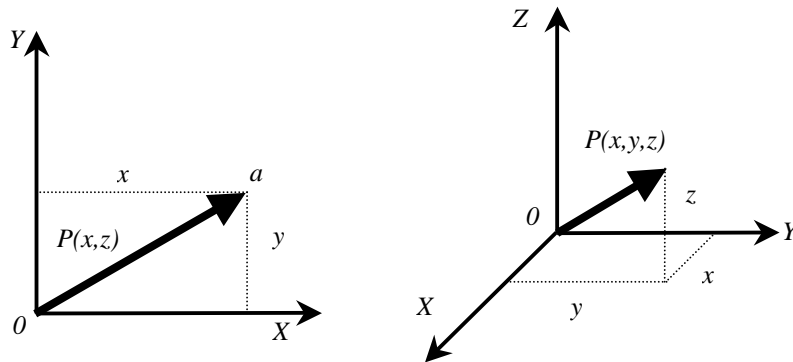
Figure 2.1: Cartesian reference system representation of: left)Two dimensions vector, right) Three dimensions vector.

right).

2.3 Orientation

A point is completely defined in the space through its position. However, for the case of a solid, it is necessary also define which is their orientation with regard to a reference system. In the case of a robot, is not enough to specify which is its end link position, but is also necessary to indicate their orientation, for example, in the case of a robot that has to kick a ball, would not be enough with specifying the points of the game-field to locate the robot appropriately, but also will be necessary to know the orientation with which the robot must kick the ball.

An orientation in the three-dimensional space is defined by three degrees of freedom or three linear-independent components. Finally, to describe the orientation of an object regarding to a reference system, it is habitual assign to the object a new system, and then study the existent space relationship between the two systems.

2.3.1 Rotation matrix

The rotation matrix is the most extended method for the description of orientations, due to the convenience that it provides by the use of the matricial algebra [21].
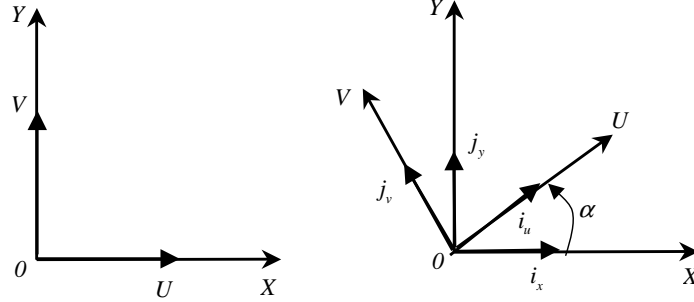
Figure 2.2: Orientation of OUV system regarding to OXY system: left) not rotated, right) rotated by a α angle.

Suppose that there are in the plane two reference systems OXY and OUV with the same origin O, being the OXY system, the fixes reference system and the OUV system, the object's mobil system (Figure 2.2 left). The unitary vectors of the coordinated axes of the OXY system are i_x, j_y while those of the OUV system are i_u, j_v [22].

A vector is represented in both systems like:

$$p_{xy} = [p_x, p_y]^T = p_x \cdot i_x + p_y \cdot j_y$$
$$[2.1]$$
$$p_{uv} = [p_u, p_v]^T = p_u \cdot i_u + p_v \cdot j_v$$

Resolving a simple series of transformations is possible to find the following equivalence:

$$\left[\begin{array}{c} p_x \\ p_y \end{array} \right] = R \left[\begin{array}{c} p_u \\ p_v \end{array} \right] [2.2]$$

where:

$$R = \left[\begin{array}{cc} i_x i_u & i_x i_v \\ j_y i_u & j_y j_v \end{array} \right] [2.3]$$

Is the so-called rotation matrix that defines the orientation of the OUV system with regard to the OXY system, and is useful to transform the coordinates of a vector in a system, to other coordinate system. Is easy to verify that this matrix is orthonormal, such that $R^{-1} = R^T$.

In two dimensions case, orientation is defined by only an independent parameter, if the relative position of the OUV system is considered, rotated by a α angle over the OXY system, figure 2.2 right). After resolve the scalar product, the R matrix will be represented by:

$$R = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} [2.4]$$

For α , where the coordinated axes of both systems coincide, the R matrix will correspond to the unitary matrix.

In a three-dimensional space, the deduction is similar. Supposed the $OXYZ$ and $OUVW$ systems are coincident in the origin, being $OXYZ$ the fixed reference system, and $OUVW$ the mobil object's reference system, whose orientation will be define, figure 2.2 left). Then, unitary vectors of the $OXYZ$ system will be i_x, j_y, k_z, while those of the $OUVW$ system will be i_u, j_v, k_w, and, a p vector on the space could be referred to any of the systems in the following way:

$$p_{uvw} = [p_u, p_v, p_w]^T = p_u \cdot i_u + p_v \cdot j_v + p_w \cdot k_w$$

$$[2.5]$$

$$p_{xyz} = [p_x, p_y, p_z]^T = p_x \cdot i_x + p_y \cdot j_y + p_z \cdot k_z$$

From the two dimensions case, this equivalence is obtained:

$$\begin{bmatrix} P_x \\ p_y \\ p_z \end{bmatrix} = R \begin{bmatrix} P_u \\ p_v \\ p_w \end{bmatrix} [2.6]$$

where:

$$R = \begin{bmatrix} i_x i_u & i_x i_v & i_x k_w \\ j_y i_u & j_y j_v & j_y k_w \\ k_z i_u & k_z j_v & k_z k_w \end{bmatrix} [2.7]$$

Is the so-called *rotation matrix* who defines the orientation of the $OUVW$ system with regard to the $OXYZ$ system, also is an orthonormal matrix, such that $R^{-1} = R^T$. The main convenience of this rotation matrix is the orientation representation of rotated systems only on a main axe of the reference system. In figure 2.2 right), is the orientation of $OUVW$ system, coincident OU axe with OX axe. This will be represented by the rotation matrix:

$$R(x, \alpha) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix} [2.8]$$
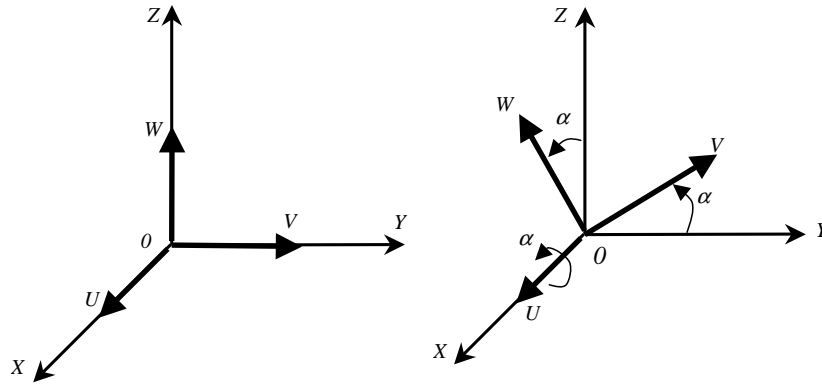
Figure 2.3: $OXYZ$ fixed reference system, and $OUVW$ object's reference system: left) both, coincident on the origin, right) OU axe coincident to OX axe.

In figure 2.4 left), the orientation of the $OUVW$ system, with OV axe coincident with OY axe, will be represented by the rotation matrix:

$$R(y, \phi) = \begin{bmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{bmatrix} [2.9]$$

In figure 2.4 right), the orientation of the $OUVW$ system, with OW axe coincident with OZ axe, will be represented by the rotation matrix:

$$R(z, \theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} [2.10]$$

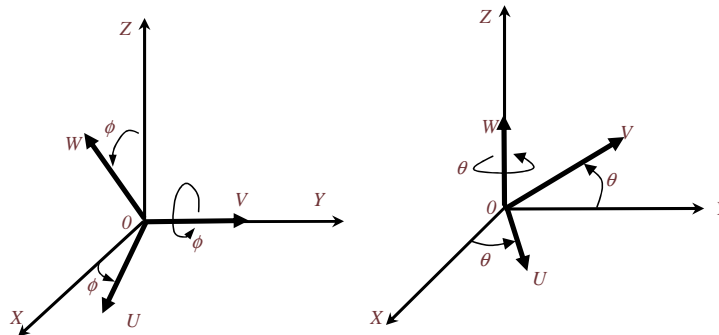Figure 2.4: Rotation of the $OUVW$ system: left) with OV axe coincident with OY axe, right) with OW axe coincident with OZ axe.

These three matrix, equations [2.8], [2.9] and [2.10], are called basic rotation matrices for a three dimensions system.

2.3.1.1 Rotation composition

The rotation matrix can be composed to express the continuous application of several rotations. Thus, if to the $OUVW$ system a rotation with a α angle is applied over OX, followed by a rotation with a ϕ angle over OY and a rotation of angle θ over OZ, the total rotation can be expressed as:

$$T = R(z,\theta), R(y,\phi), R(x,\alpha) =$$
$$\begin{bmatrix} C\theta & -S\theta & 0 \\ S\theta & C\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} C\phi & 0 & S\phi \\ 0 & 1 & 0 \\ -S\phi & 0 & C\phi \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & C\alpha & -S\alpha \\ 0 & S\alpha & C\alpha \end{bmatrix}$$

$$= \begin{bmatrix} C\theta C\phi & -S\theta C\alpha + C\theta S\phi S\alpha & S\theta S\alpha + C\theta S\phi C\alpha \\ S\theta C\phi & C\theta C\alpha + S\theta S\phi S\alpha & -C\theta S\alpha + S\theta S\phi C\alpha \\ -S\phi & C\phi S\alpha & C\phi C\alpha \end{bmatrix} [2.11]$$

Where $C\theta$ express $cos\theta$ and $S\theta$ express $sin\theta$.

Is important to consider the order in which the rotations will be performance, because the matrix product is not commutative. Thus, if the rotation were first perform a θ angle over OZ, followed by a rotation with a ϕ angle over OY and a rotation of angle α over OX, the global rotation can be expressed as:

$$T = R(x,\alpha), R(y,\phi) R(z,\theta), =$$
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & C\alpha & -S\alpha \\ 0 & S\alpha & C\alpha \end{bmatrix} \begin{bmatrix} C\phi & 0 & S\phi \\ 0 & 1 & 0 \\ -S\phi & 0 & C\phi \end{bmatrix} \begin{bmatrix} C\theta & -S\theta & 0 \\ S\theta & C\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} C\theta C\phi & -C\phi S\theta & S\phi \\ S\alpha S\phi C\theta + C\alpha S\theta & -S\alpha S\phi S\theta + C\alpha C\theta & -S\alpha C\phi \\ -C\alpha S\phi C\theta + S\alpha S\theta & C\alpha S\phi S\theta + S\alpha C\theta & C\alpha C\phi \end{bmatrix} [2.12]$$

As can be note, this matrix differs from the previous one.

2.4 Homogeneous transformation matrix [HTM]

In the previous sections, some methods have been explained to represent the position or the orientation of a solid in the space. But none of these methods

by itself allows a combined representation of position and orientation (localization). In order to solve this problem, the denominated homogeneous coordinate transformation matrix or homogeneous transformation matrix is introduced.

2.4.1 Coordinates

The representation by homogeneous coordinates for the solid localization in a *n*-dimensional space is achieved through coordinates of a *(n+1)*-dimensional space. In other words, a *n*-dimensional space is represented by homogeneous coordinated in *(n+1)*-dimensions, in such a way that a *p(x,y,z)* vector will come represented by *p(wx,wy,wz,w)*, where *w* has an arbitrary value and it represents a scale factor [23].

In general, a vector *p = ai + bj + ck*, where *i, j* and *k* are the unitary vectors of *OX, OY* and *OZ* axes of *OXYZ* reference system, is represented in homogeneous coordinates by the column vector:

$$p = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} ax \\ by \\ cz \\ w \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \\ 1 \end{bmatrix} \quad [2.13]$$

For example, vector *2i +3j + 4k* can be represented in homogeneous coordinates as $[2,3,4,1]^T$ or by $[4,6,8,2]^T$ or as $[-6,-9,-12,-3]^T$, etc. The null vectors are represented as $[0,0,0,n]^T$ where *n* is not-null. The vectors in the $[a,b,c,0]^T$ format are useful to represent addresses, because they represent vectors of infinite longitude.

From the homogeneous coordinates definition is take the concept of homogeneous transformation matrix. A *T* homogeneous transformation matrix is a 4 x 4 dimensions matrix and represents the transformation of a homogeneous coordinates vector from a coordinate system to another coordinate system.

$$T = \begin{bmatrix} R_{3x3} & P_{3x1} \\ f_{1x3} & w_{1x1} \end{bmatrix} = \begin{bmatrix} Rotation & Translation \\ Perspective & Scale \end{bmatrix} \quad [2.14]$$

A homogeneous matrix is composed by four sub-matrices of different size: a sub-matrix $R_{3\times3}$ who corresponds to a *rotation matrix*; a sub-matrix $P_{3\times1}$ who corresponds to a *translation vector*; a sub-matrix $f_{1\times3}$ who represents the *perspective transformation*, and a sub-matrix $W_{1\times1}$ who represents a *global scale*. In robotics, is important to know only the $R_{3\times3}$ and $P_{3\times1}$ value, and considering

the components null and unitary for $W_{1\times1}$. Since this is a 4x4 matrix, the used vectors will have 4 dimensions (that will be the homogeneous coordinates of the three-dimensional vector) .

2.4.2   Application

As mentioned, if the perspective transformation value is considered null and the global scale value is unitary, the *homogeneous transformation matrix* T will be represented as:

$$T = \left[\begin{array}{cc} R_{3x3} & P_{3x1} \\ 0 & 1 \end{array} \right] = \left[\begin{array}{cc} Rotation & Translation \\ 0 & 1 \end{array} \right] [2.15]$$

Who represents the orientation and position of a OUVW system rotated and translated with regard to $OXYZ$ reference system. This matrix is useful to find the (r_x, r_y, r_z) coordinates of the r vector on $OXYZ$ system from its (r_u, r_v, r_w) coordinates on the $OUVW$:

$$\left[\begin{array}{c} r_x \\ r_y \\ r_z \\ 1 \end{array} \right] = T \left[\begin{array}{c} r_u \\ r_v \\ r_w \\ 1 \end{array} \right] [2.16]$$

Also it can be used to represent the rotation and translation of a vector with regarding to a $OXYZ$ fixed reference system, thus, a r_{xyz} vector rotated according to $R_{3\times3}$ and translated according to $P_{3\times1}$ becomes in a r'_{xyz} vector, represented by:

$$\left[\begin{array}{c} r'_x \\ r'_y \\ r'_z \\ 1 \end{array} \right] = T \left[\begin{array}{c} r_x \\ r_y \\ r_z \\ 1 \end{array} \right] [2.17]$$

In summary, a *homogeneous transformation matrix* is used to:

1. Represent the position and orientation of a rotated and translated system in $OUVW$ with regard to a fixed reference system $OXYZ$.

2. Transform a vector expressed in coordinated with regard to a $OUVW$ system, to a $OXYZ$ reference system.

3. Rotate and translate a vector with regard to a fixes $OXYZ$ reference system.

Translation

Suppose that the $OUVW$ reference system is translated only by a $p = p_x i + p_y j + p_z k$ vector with regard to the $OXYZ$ system. Then the T matrix will correspond to a homogeneous translation matrix, also called *basic translation matrix* [24]:

$$T(p) = \begin{bmatrix} 1 & 0 & 0 & p_x \\ 0 & 1 & 0 & p_y \\ 0 & 0 & 1 & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad [2.18]$$

Any r vector, represented in the $OUVW$ system by r_{uvw}, will have as a components of the vector with regard to the system $OXYZ$:

$$\begin{bmatrix} r_x \\ r_y \\ r_z \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & p_x \\ 0 & 1 & 0 & p_y \\ 0 & 0 & 1 & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_u \\ r_v \\ r_w \\ 1 \end{bmatrix} = T \begin{bmatrix} r_u + p_x \\ r_v + p_y \\ r_w + p_z \\ 1 \end{bmatrix} \quad [2.19]$$

and a r_{xyz} vector displaced according to T will have as a r'_{xyz} components:

$$\begin{bmatrix} r'_x \\ r'_y \\ r'_z \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & p_x \\ 0 & 1 & 0 & p_y \\ 0 & 0 & 1 & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_x \\ r_y \\ r_z \\ 1 \end{bmatrix} = T \begin{bmatrix} r_x + p_x \\ r_y + p_y \\ r_z + p_z \\ 1 \end{bmatrix} \quad [2.20]$$

Translation (calculus example)

According to the figure 2.5, the $O'UVW$ system is translated a *p(6,-3,8)* vector with regarding to the $OXYZ$ system.

To calculate the coordinates (r_x, r_y, r_z) of the r vector, whose coordinates with regard to the system $O'UVW$ are r_{uvw} *(-2, 7, 3)*, the equation [2.19] must be applied:

$$\begin{bmatrix} r_x \\ r_y \\ r_z \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 8 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ 7 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ 11 \\ 1 \end{bmatrix}$$

Now, to calculate the r'_{xyz} vector (Figure 2.6), who result from translate the $r_{xyz}(4,4,11)$ vector according to the $T(p)$ transformation with p *(6,-3,8)* the equation [2.20] must be be applied:
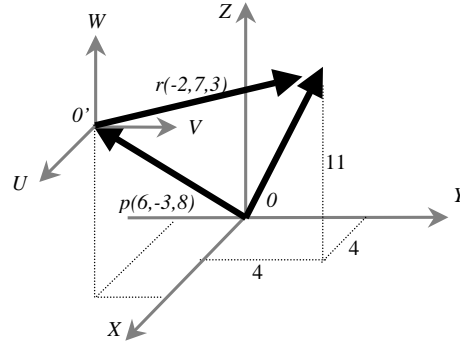
Figure 2.5: $O'UVW$ system translated a p vector with regarding to the $OXYZ$ system.
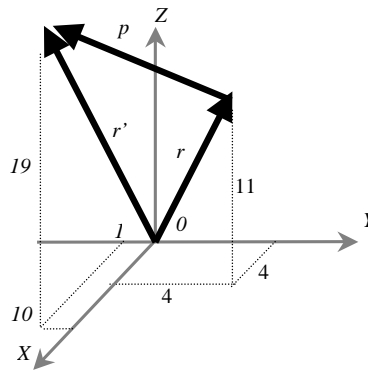

Figure 2.6: r'_{xyz} vector resulted from the translation of r_{xyz}.

$$\begin{bmatrix} r'_x \\ r'_y \\ r'_z \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 8 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 4 \\ 11 \\ 1 \end{bmatrix} = \begin{bmatrix} 10 \\ 1 \\ 19 \\ 1 \end{bmatrix}$$

Rotation

If the $OUVW$ system is rotated only with regard to the $OXYZ$ system, then the rotation sub-matrix R defines the rotation, and it belongs to the type of rotation matrix presented in the 2.2.1 section. Like in that section, three basic homogeneous matrix of rotations can be defined according to a rotation made

on one of the three coordinated axes OX, OY or OZ of the $OXYZ$ reference system.

$$T(x, \alpha) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha & 0 \\ 0 & \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad [2.21]$$

$$T(y, \phi) = \begin{bmatrix} \cos\phi & 0 & \sin\phi & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\phi & 0 & \cos\phi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad [2.22]$$

$$T(z, \theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad [2.23]$$

Any r vector, represented in the rotated $OUVW$ system for r_{uvw} will have as components (r_x, r_y, r_z) on the $OXYZ$ system:

$$\begin{bmatrix} r_x \\ r_y \\ r_z \\ 1 \end{bmatrix} = T \begin{bmatrix} r_u \\ r_v \\ r_w \\ 1 \end{bmatrix} \quad [2.24]$$

And a r_{xyz} vector rotated according to T will be represented by r'_{xyz} as:

$$\begin{bmatrix} r'_x \\ r'_y \\ r'_z \\ 1 \end{bmatrix} = T \begin{bmatrix} r_x \\ r_y \\ r_z \\ 1 \end{bmatrix} \quad [2.25]$$

Rotation (calculus example)

According to the figure 2.7, the $OUVW$ system is rotated at $-90°$ around of the OZ axe, with regarding to $OXYZ$ system. To calculate the coordinates (r_x, r_y, r_z) of the r vector, whose coordinated with regard to the system $O'UVW$ are $r_{uvw} = [4, 8, 12]^T$, the equation [2.24] must be applied:
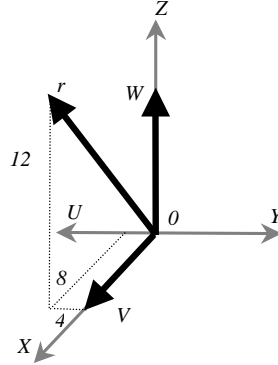
Figure 2.7: $OUVW$ system rotated $-90°$ around of the OZ axe, with regarding to $OXYZ$ system.

$$
\begin{bmatrix} r_x \\ r_y \\ r_z \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 8 \\ 12 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ -4 \\ 12 \\ 1 \end{bmatrix} [2.26]
$$

Translation-rotation

The main advantage of the homogeneous matrix is its capacity to represent at the same time, the position and orientation (localization). This representation is performed by using the rotation matrix $R_{3\times3}$ and the translation vector $P_{3\times1}$ at the same time in a single homogeneous transformation matrix.

Translation and rotation are transformations that are performed in relation to a reference system. Therefore, if the position and orientation of a $O'UVW$ system will be represented, who originally is coincident with the reference system but has been rotated and translated, thus will be necessary to know, if the rotation has been performed first and then a translation or vice versa, because these space transformations are not commutative. In Figure 2.8, not-commutativity is presented graphically. The origin is a $OUVW$ system coincident with $OXYZ$ in which a translation according to a p_{xyz} vector is applied, then a rotation of $180°$ around the OZ axis. If first, a rotation is made and then a translation is applied , the obtained final system is $OU'V'W'$.

On the other hand, if first the translation is made and then the rotation is applied, the obtained system is $OU"V"W"$, who represents a completely different localization with regard to the previous final system.
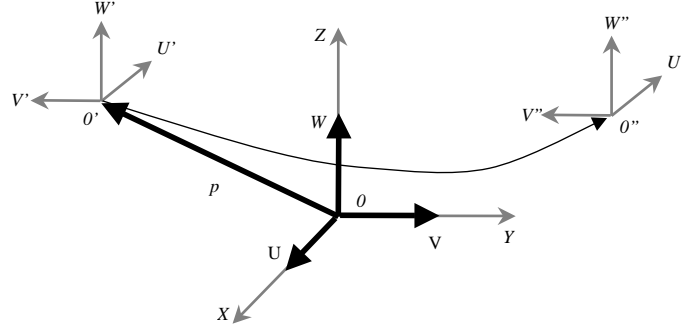
Figure 2.8: Obtaining different final systems, according to the transformations order.

Therefore, different homogeneous matrix will be used according to which operation (rotation or a translation) is made at first.

Rotation followed by translation

If first, a rotation is made over a coordinate axe of $OXYZ$ system followed by a translation, then the homogeneous matrix will be expressed as:

a) Rotation of a α angle over OX axis followed by a p_{xyz} translation vector.

$$T((x,\alpha),p) = \begin{bmatrix} 1 & 0 & 0 & p_x \\ 0 & \cos\alpha & -\sin\alpha & p_y \\ 0 & \sin\alpha & \cos\alpha & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad [2.27]$$

b) Rotation of a ϕ angle over OY axis followed by a p_{xyz} translation vector.

$$T((y,\phi),p) = \begin{bmatrix} \cos\phi & 0 & \sin\phi & p_x \\ 0 & 1 & 0 & p_y \\ -\sin\phi & 0 & \cos\phi & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad [2.28]$$

c) Rotation of a θ angle over OZ axis followed by a p_{xyz} translation vector.

$$T((z,\theta),p) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & p_x \\ \sin\theta & \cos\theta & 0 & p_y \\ 0 & 0 & 1 & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad [2.29]$$
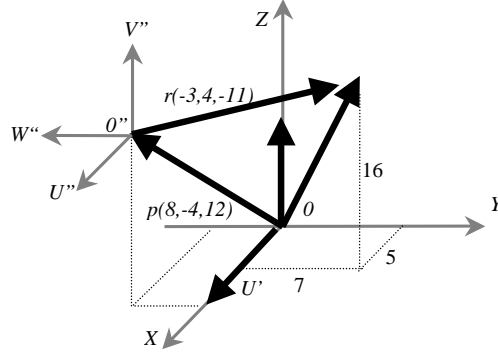
Figure 2.9: $OUVW$ system rotated 90° around the OX axis and translated a p (8,-4,12) vector, which regard to $OXYZ$ system.

Rotation followed by translation (calculus example)

A $OUVW$ system has been rotated 90° around the OX axis, and then translated a $p(8,-4,12)$, vector which regard to $OXYZ$ system (figure 2.9). To calculate the (r_x, r_y, r_z) coordinates of the r vector with $r_{uvw} = (-3, 4, -11)$ coordinates, the equation [2.27] must be used:

$$
\begin{bmatrix} r_x \\ r_y \\ r_z \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 8 \\ 0 & 0 & -1 & -4 \\ 0 & 1 & 0 & 12 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -3 \\ 4 \\ -11 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \\ 16 \\ 1 \end{bmatrix}
$$

Translation followed by rotation

If first, a translation is made, followed by a rotation over a coordinate axe of $OXYZ$ system, the homogeneous matrix are:

a) Translation of a p_{xyz} vector, followed by a α rotation over OX axis.

$$
T(p,(x,\alpha)) = \begin{bmatrix} 1 & 0 & 0 & p_x \\ 0 & \cos\alpha & -\sin\alpha & p_y\cos\alpha - p_z\sin\alpha \\ 0 & \sin\alpha & \cos\alpha & p_y\sin\alpha + p_z\cos\alpha \\ 0 & 0 & 0 & 1 \end{bmatrix} [2.30]
$$

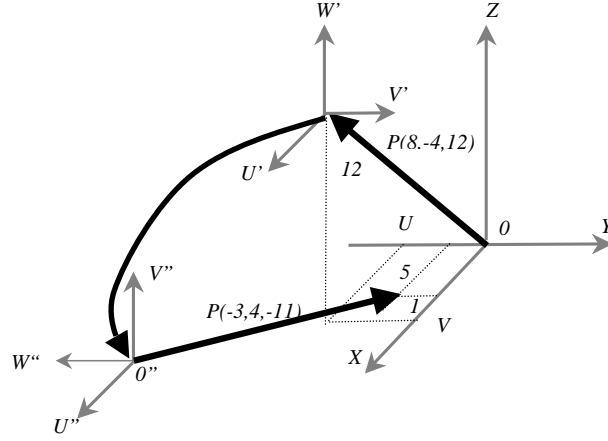b) Translation of a p_{xyz} vector, followed by a ϕ rotation over OY axis.

Figure 2.10: $OUVW$ system translated a $p(8,-4,12)$ vector, which regard to the $OXYZ$ system, and rotated 90° over the OX axis.

$$T(p,(y,\phi)) = \begin{bmatrix} \cos\phi & 0 & \sin\phi & p_x\cos\phi + p_z\sin\phi \\ 0 & 1 & 0 & p_y \\ -\sin\phi & 0 & \cos\phi & p_z cos\phi - p_x\sin\phi \\ 0 & 0 & 0 & 1 \end{bmatrix} [2.31]$$

c) Translation of a p_{xyz} vector, followed by a θ rotation over OZ axis.

$$T(p,(z,\theta)) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & p_x\cos\theta - p_y\sin\theta \\ \sin\theta & \cos\theta & 0 & p_x\sin\theta + p_y\cos\theta \\ 0 & 0 & 1 & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} [2.32]$$

Translation followed by rotation (calculus example)

If the $OUVW$ system is translated a $p(8,-4,12)$ vector, which regard to the $OXYZ$ system (figure 2.10) and rotated 90° over the OX axis. Then, to calculate the (r_x, r_y, r_z) coordinates of the r vector of $r_{uvw} = (-3, 4, -11)$ coordinates, the equation [2.30] must be used:

$$\begin{bmatrix} r_x \\ r_y \\ r_z \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 8 \\ 0 & 0 & -1 & -12 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -3 \\ 4 \\ -11 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

2.4.3 Geometric meaning

As already has been described, a homogeneous matrix is useful to transform a vector expressed in homogeneous coordinates in a $O'UVW$ system, to its expression in coordinates of the $OXYZ$ reference system. Also can be used to rotate and to spin a vector referred to a fixed reference system, and in definitive is useful to express the orientation and position of a $O'UVW$ reference system with regard to a fixed $OXYZ$ system.

The T transformation matrix is usually written in the following way [25]:

$$T = \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} n & o & a & p \\ 0 & 0 & 0 & 1 \end{bmatrix} [2.33]$$

Where n,o,a are orthonormal and they represents the orientation, and p is a vector which represents the position.

If a vector $r_{uvw} = [0,0,0,1]^T$ is considered, (the origin of the $O'UVW$ system) the application of the T matrix, that represents the transformation (translation + rotation) of the $O'UVW$ system, with regard to $OXYZ$, then, r_{xyz} can be obtained as:

$$T = \begin{bmatrix} n & o & a & p \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} p_x \\ p_y \\ p_z \\ 1 \end{bmatrix} [2.34]$$

Which coincides with the p column vector of T. Therefore, this column vector, represent the position of the origin of $O'UVW$, with regarding to the $OXYZ$ system.

If the $[1,0,0,1]^T$ homogeneous vector is considered with regarding to the $OUVW$ coordinate system (the vector on the coordinate $O'U$ axis of the $O'UVW$ system) and supposing the translation p, a null vector, then 2.35 can be obtained:

$$\begin{bmatrix} r_x \\ r_y \\ r_z \\ 1 \end{bmatrix} = \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} n_x \\ n_y \\ n_z \\ 1 \end{bmatrix} [2.35]$$

That means, that the column vector n represents the coordinates of the $O'U$ axis of the $O'UVW$ system with regarding the system $OXYZ$.

Similar, if a transformation of the $[0, 1, 0, 1]^T$ and $[0, 0, 1, 1]^T$ vectors referred to the $O'UVW$ system is performed, then column vector o represents the coordinates of the $O'V$ axis of the $O'UVW$ system with regarding the $OXYZ$ system, and that the column vector a represents the coordinates of the $O'W$ axis of the $O'UVW$ system with regarding the $OXYZ$ system.

Consequently, the vector *n, o* and *a* define an orthonormal to the right, that means:

$$|n| = |o| = |a| = 1$$

$$n \times o = a$$

The rotation sub-matrix $[n, o, a]$ corresponds to an orthonormal matrix:

$$\begin{bmatrix} n & o & a \end{bmatrix}^{-1} = \begin{bmatrix} n & o & a \end{bmatrix}^{T}$$

The inverse matrix of the homogeneous transformation matrix T is easily attainable, and corresponds to the following expression:

$$T^{-1} = \begin{bmatrix} n_x & n_y & n_z & -n^T p \\ o_x & o_y & o_z & -o^T p \\ a_x & a_y & a_z & -a^T p \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad [2.36]$$

If the relationship $r_{xyz} = T r_{uvw}$ is multiplied in both members by T^{-1}, than:

$$T^{-1} r_{xyz} = r_{uvw}$$

Performing the previous process, can be deduced, that the row vectors of the rotation sub-matrix T (column vectors of the rotation sub-matrix of T^{-1}), represent the main axes of the $OXYZ$ reference coordinate system, with regard to $OUVW$. That is, the row vectors of the $[n \ o \ a]$ matrix represent another orthonormal to right.

2.4.4   Composition

As previously has been mention, the homogeneous matrix transformation is useful to represent the rotation and the translation over a reference system. This characteristic of the homogeneous matrix is even more important when the homogeneous matrix is composed to describe several rotations and translations over a determined reference system.

Thus, a complex transformation can be break down in several simple transformations (basic rotations and translations).

Operations performed over OXYZ fixed system

For example, to represent a α angle rotation over OX axis, followed by a ϕ angle rotation over OY axis and a rotation of θ angle over OZ axis, the composition of this basic rotation matrix can be used:

$$T = T(z, \theta)T(y, \phi)T(x, \alpha)$$

$$T = \begin{bmatrix} C\theta & -S\theta & 0 & 0 \\ S\theta & C\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} C\phi & 0 & S\phi & 0 \\ 0 & 1 & 0 & 0 \\ -S\phi & 0 & C\phi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & C\alpha & -S\alpha & 0 \\ 0 & S\alpha & C\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} C\theta C\phi & -S\theta C\alpha + C\theta S\phi S\alpha & S\theta S\alpha + C\theta S\phi C\alpha & 0 \\ S\theta C\phi & C\theta C\alpha + S\theta S\phi S\alpha & -C\theta S\alpha + S\theta S\phi C\alpha & 0 \\ -S\phi & C\phi S\alpha & C\phi C\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Because matrix product is not commutative, neither transformation matrix composition. Then, if application's order of the transformations is changed, logically, the result will be different, for example:

$$T = T(x, \alpha)T(y, \phi)T(z, \theta)$$

$$T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & C\alpha & -S\alpha & 0 \\ 0 & S\alpha & C\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} C\phi & 0 & S\phi & 0 \\ 0 & 1 & 0 & 0 \\ -S\phi & 0 & C\phi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} C\theta & -S\theta & 0 & 0 \\ S\theta & C\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T = \begin{bmatrix} C\theta C\phi & -C\phi S\theta & S\phi & 0 \\ S\alpha S\phi C\theta + C\alpha S\theta & -S\alpha S\phi S\theta + C\alpha C\theta & -S\alpha C\phi & 0 \\ -C\alpha S\phi C\theta + S\alpha S\theta & C\alpha S\phi S\theta + S\alpha C\theta & C\alpha C\phi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Calculus example (OXYZ fixed system)

The following matrix multiplication order will be necessary to obtain T (the transformation matrix) who represents the $OUVW$ system obtained by a $-90°$ turn over OX axis, a $p_{xyz} = (5, 5, 10)$ vector translation, and a $90°$ turn over OZ, (all at the $OXYZ$ system):

$$T = T(z, 90°)T(p)T(x, -90°)$$

$$T = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 10 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T = \begin{bmatrix} 0 & 0 & -1 & -5 \\ 1 & 0 & 0 & 5 \\ 0 & -1 & 0 & 10 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Operations performed over the $OUVW$ moving system

Also is possible to compose matrix transformations referred to the moving system, to achieved, will be enough to perform the matrix operations in inverse order. For example:

$$T = T(x, \alpha)T(v, \phi)T(w, \theta)$$

$$T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & C\alpha & -S\alpha & 0 \\ 0 & S\alpha & C\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} C\phi & 0 & S\phi & 0 \\ 0 & 1 & 0 & 0 \\ -S\phi & 0 & C\phi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} C\theta & -S\theta & 0 & 0 \\ S\theta & C\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T == \begin{bmatrix} C\theta C\phi & -C\phi S\theta & S\phi & 0 \\ S\alpha S\phi C\theta + C\alpha S\theta & -S\alpha S\phi S\theta + C\alpha C\theta & -S\alpha C\phi & 0 \\ -C\alpha S\phi C\theta + S\alpha S\theta & C\alpha S\phi S\theta + S\alpha C\theta & C\alpha C\phi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

is necesarry, to obtain a matrix that represents a α angle turn over OX axis (at the fixed $OXYZ$ system) followed by a ϕ turn over OV axis and a θ angle turn over OW axis (at $OUVW$ system in movement).

Calculus example ($OUVW$ moving system)

The following matrix multiplication steps will be necessary to obtain the transformation matrix, which represents the following transformations over a fixed $OXYZ$ system: translation of a $p_{xyz} = (-3, 10, 10)$ vector, $-90°$ turn over the OU axis, and a $90°$ turn over the OV axis:

$$T = T(p)T(u, -90°)T(v, 90°)$$

$$T = \begin{bmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 10 \\ 0 & 0 & 1 & 10 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T = \begin{bmatrix} 0 & 0 & 1 & -3 \\ -1 & 0 & 0 & 10 \\ 0 & -1 & 0 & 10 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Summarize, when a composition of diverse transformations by homogeneous matrix is implemented, the followings points must be keep in mind :

1. When the $OXYZ$ fixed system and the $OUVW$ transformed system are coincident, the homogeneous transformation matrix will be a 4 x 4 identity matrix.

2. If the $OUVW$ system is obtained by rotations and translations defined with regard to the fixed $OXYZ$ system, the homogeneous matrix that represents each transformation will be pre-multiply with the previous transformations matrix.

3. If the $OUVW$ system is obtained by rotations and translations defined with regard to the mobile system, the homogeneous matrix that represents each transformation will be post-multiply by the previous matrices transformations.